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Gauge-string duality for superconformal deformations of $N = 4$ Super Yang-Mills theory

Sergey A. Frolov,[a*] **Radu Roiban**[b] **and Arkady A. Tseytlin**[c†]

[a] *Department of Mathematics/Sciences, SUNY IT*
P.O. Box 3050, Utica, NY 13504-3050, U.S.A.
[b] *Department of Physics, Princeton University*
Princeton, NJ 08544, U.S.A.
[c] *Department of Physics, The Ohio State University*
Columbus, OH 43210, U.S.A.
E-mail: frolovs@sunyit.edu, rroiban@princeton.edu, tseytlin@mps.ohio-state.edu

Abstract: We analyze in detail the relation between an exactly marginal deformation of $\mathcal{N} = 4$ SYM — the Leigh-Strassler or "β-deformation" — and its string theory dual (recently constructed in hep-th/0502086) by comparing energies of semiclassical strings to anomalous dimensions of gauge-theory operators in the two-scalar sector. We stress the existence of integrable structures on the two sides of the duality. In particular, we argue that the integrability of strings in $AdS_5 \times S^5$ implies the integrability of the deformed world sheet theory with real deformation parameter. We compare the fast string limit of the worldsheet action in the sector with two angular momenta with the continuum limit of the coherent state action of an anisotropic XXZ spin chain describing the one-loop anomalous dimensions of the corresponding operators and find a remarkable agreement for all values of the deformation parameter. We discuss some of the properties of the Bethe Ansatz for this spin chain, solve the Bethe equations for small number of excitations and comment on higher loop properties of the dilatation operator. With the goal of going beyond the leading order in the 't Hooft expansion we derive the analog of the Bethe equations on the string-theory side, and show that they coincide with the thermodynamic limit of the Bethe equations for the spin chain. We also compute the $1/J$ corrections to the anomalous dimensions of operators with large R-charge (corresponding to strings with angular momentum J) and match them to the 1-loop corrections to the fast string energies. Our results suggest that the impressive agreement between the gauge theory and semiclassical strings in $AdS_5 \times S^5$ is part of a larger picture underlying the gauge/gravity duality.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence.

[*] Also at Steklov Mathematical Institute, Moscow.
[†] Also at Imperial College London and Lebedev Institute, Moscow.

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1. Introduction

A relation between large-N gauge theory and string theory has been a subject of active investigation for more than three decades. The advent of the AdS/CFT correspondence [1–3] provided us with new concepts and new tools to try to put this remarkable relation on a firm ground. Still, understanding the $AdS_5 \times S^5$ — $\mathcal{N}=4$ SYM duality in detail beyond the BPS and near BPS [4] limit remains a challenge.

It was suggested in [5, 6] that concentrating on semiclassical states with large quantum number (e.g., spin) may allow one to verify the agreement between the large spin dependence of the energies E of strings in AdS_5 and of the anomalous dimensions Δ of the corresponding SYM operators.[1] Moreover, it was proposed in [8] that there should exist a large class of multi-spin semiclassical string states and dual "long" SYM operators for which the leading coefficient functions in the large spin expansion of E and Δ may match exactly. This was indeed confirmed explicitly, first on particular examples [9 − 14] and then in general [15 − 18] (see also [19] for a review and additional references).

The underlying reason for this agreement (which may look unexpected in view of the different limits taken on the string and the gauge theory sides) appears to be the equivalence of the two integrable systems that govern the corresponding leading-order corrections and whose structure happens to be tightly constrained. A conceptually simple and universal way of understanding this is by showing the equivalence of (i) the "fast-string" limit of the string action producing a non-relativistic sigma model action for "slow" string degrees of freedom, and (ii) the "Landau-Lifshitz" action for coherent spin chain states describing the relevant sector (semiclassical spin wave states with energies scaling as $1/J$ in the large spin J limit) of the ferromagnetic spin chain appearing on the gauge theory side [15, 16].

Both the world sheet theory of strings in $AdS_5 \times S^5$ and the dilatation operator of the planar $\mathcal{N} = 4$ theory are described by integrable systems, the latter being an integrable spin chain [20, 21].[2] One can utilize the integrability to show that the integral equation and the associated spectral curve that classifies classical solutions of the string sigma model (expanded to first two orders in the "fast-string" parameter $\frac{\lambda}{J^2}$) is equivalent to the "thermodynamic" (large J) limit of the one- and two-loop Bethe ansatz equations appearing on the gauge theory side [17, 18]. This relation between the classical "string Bethe equation" and the thermodynamic limit of the spin chain Bethe ansatz (which appears to extend also to the first subleading $1/J$ level [27]) provides an alternative demonstration of the "$E = \Delta$" equivalence. It also prompts to try to find an analog of an interpolating Bethe ansatz that would describe the spectrum of free strings in $AdS_5 \times S^5$ or of the corresponding SYM operators at any finite 't Hooft coupling λ [28, 29].

Given this remarkable progress in understanding $AdS_5 \times S^5$ string — $\mathcal{N} = 4$ SYM duality one would like to try to extend it to more "realistic", i.e. less supersymmetric, examples, anticipating that the semiclassical string states should be capturing some general features of the correspondence between planar gauge theory and free string theory. It is natural to start with examples where the gauge theory side is superconformal at *any* value of the 't Hooft coupling λ. In such a case one would have, as in $\mathcal{N} = 4$ SYM, a luxury of being able to use the gauge theory perturbative expansion at small λ and the string theory perturbative expansion at large λ and then to compare the results for semiclassical states with large J in an appropriate scaling limit. One obvious class of such examples are orbifolds of the

[1] See [7] for recent progress in this direction.

[2] This spin chain is explicitly known at the 1-loop level in the full theory [22], to the first few loop orders in various subsectors closed under renormalization group flow [23], and the asymptotic Bethe ansatz for the $su(2)$ subsector were proposed in [24] (see [25] for a review and references). For dilatation operators and associated integrable spin chains in less supersymmetric gauge theories see also [26] and references therein.

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$AdS_5 \times S^5$ string dual to orbifolds of $\mathcal{N} = 4$ SYM [30] (see [31]) but they are somewhat trivial not involving a new continuous parameter: we would like to have an adjustable *continuous* deformation parameter which could allow us to define new scaling limits.

A non-trivial starting point would be the well-known one-parameter family of $\mathcal{N} = 1$ gauge theories which are exactly marginal deformations [33] of the $\mathcal{N} = 4$ SYM theory (for earlier work on $\mathcal{N} = 1$ superconformal theories see [32]), but until very recently [34] the corresponding dual type-IIB supergravity background and thus the dual superstring theory was not known explicitly.[3] The remarkable observation made in [34] is that in the case of a real deformation parameter β the dual 1/4 supersymmetric supergravity background can be constructed by applying a combination of T-duality, shift of angle and another T-duality ("TsT"-transformation) to the original $AdS_5 \times S^5$ background.[4] The resulting string theory in this background, whose metric is a product of AdS_5 and a 5-space with topology of S^5 and which depends, in addition to the radius $R = \sqrt{\alpha' \sqrt{\lambda}}$, also on the new deformation parameter β, may then be viewed as an exactly marginal deformation (now in the 2d sense) of the $AdS_5 \times S^5$ superstring theory. One may also try to generalize to the case of the non-zero imaginary part of the parameter β by applying an extra S-duality transformation to the TsT transformed $AdS_5 \times S^5$ supergravity background. However, in this case the supergravity background may receive non-trivial α'-corrections and thus the corresponding string theory may not be possible to define in a useful closed form (see also below).

Our aim in this paper will be to study the duality example of [34] by comparing semi-classical string states to a class of gauge theory scalar operators whose 1-loop anomalous dimensions are again described by an integrable spin chain [37, 38]. Surprisingly, we will find that many of the results that were obtained in the undeformed $\beta = 0$ case of $AdS_5 \times S^5$ — $\mathcal{N} = 4$ SYM duality have a straightforward generalization to the case of (real) non-zero β. There is again an equivalence between the "fast-string" limit of the classical string action and the Landau-Lifshitz action associated to the 1-loop spin chain, i.e. the matching between leading order terms in the string energies and anomalous dimensions of the dual gauge theory operators. There is also an analog of classical "string Bethe equations" of [17].

We shall begin in section 2 by recalling the form of the exactly marginal deformations of the $\mathcal{N} = 4$ SYM [33] and their gravity duals [34]. We shall emphasize the string theory consequences of the T-duality algorithm used to construct them.

To be able to compare string theory and gauge theory results it is important to understand the relation between the parameters appearing in the two theories and their scaling properties in the sector of states with large quantum numbers. We will discuss this in section 3.

One of the consequences of the TsT transformation used to find the string theory dual of the deformed gauge theory is that for each semiclassical string solution in $AdS_5 \times S^5$ there exists the corresponding one in the deformed background.[5] In section 4 we shall first present

[3] A perturbative approach to finding the corresponding supergravity background was developed in [35].

[4] Similar transformations were used in [36] to construct Melvin-type magnetic flux tube backgrounds from flat space and also to solve the corresponding string theory.

[5] For integer shift parameter the TsT transformation is a symmetry of the $AdS_5 \times S^5$ string theory, i.e. while the two supergravity backgrounds still look very different, the corresponding string spectra must be

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an example of such a solution and then follow [16, 40] to find an effective action describing all such solutions in the large angular momentum limit. The corresponding equation of motion expanded to the leading order in large angular momentum is a “twisted” version of the (anisotropic for complex β) Landau-Lifshitz equation which is known to be integrable.

Next, we shall turn in section 5 to the $\mathcal{N} = 1$ supersymmetric gauge theory side, starting with the spin chain description of the 1-loop dilatation operator in the deformed gauge theory in a scalar field sector [37, 38]. We shall then analyze the continuum limit of the coherent state expectation value of the integrable spin chain hamiltonian and construct the corresponding Landau-Lifshitz effective action for low wave-length spin excitations. As in the undeformed case, this action happens to be exactly the same as found on the string side. This implies matching between string energies and 1-loop gauge theory anomalous dimensions and also matching of the associated integrable structures.

In the following section we shall present the Bethe ansatz for the integrable spin chain representing the 1-loop dilatation operator of the deformed theory. We shall then explicitly solve the Bethe equations in the small excitation (or near-BPS or “small string”) limit, i.e. in the regime corresponding to a particular plane wave limit of the geometry [34]. We shall also discuss higher-loop corrections and point out that while the 1-loop dilatation operator in the 2-scalar sector is integrable for generic deformation, the 2-loop correction preserves integrability property only for real deformations.

In section 7, with a motivation to go beyond the leading order, we will show how the derivation of the string Bethe equation describing classical solutions of strings moving on S^3 [17] generalizes to the case of real β deformation. We shall use the Lax representation constructed in [41] which implies integrability of strings moving in the background of [34]. This integrability property is remarkable since the target space of the deformed sigma model is no longer a coset space.

In section 8 we shall compute, following the discussion of $\beta = 0$ case in [27], the 1-loop subleading $1/J$ corrections in the 2-spin sector and demonstrate that the correspondence between the gauge and string theory results holds also in the deformed theory.

Our conclusions will be summarized in section 9.

In appendix A we shall discuss the relation between the spin chain hamiltonian used in this paper and a more general one considered in [37]. In appendix B we shall give some details on asymptotics of the monodromy matrix used in section 7.

2. Deformed $\mathcal{N} = 4$ SYM and its string theory dual

The $\mathcal{N} = 4$ super-Yang-Mills theory exhibits two exactly marginal deformations [33]. The corresponding terms in the superpotential in the $\mathcal{N} = 1$ superspace action are $\mathrm{Tr}[\Phi_1\{\Phi_2, \Phi_3\}]$ and $\sum_{i=1}^{3} \mathrm{Tr}[\Phi_i^3]$. In the following we will be interested only in the first type of deformation.

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isomorphic (cf. [39]). This is true also in the case when one performs shifts of the 3 isometric angles of S^5 by 2 integer parameters.

2.1 β-deformed $\mathcal{N}=4$ SYM theory: a brief review

In the notation of [34], the superpotential of $\mathcal{N}=4$ SYM theory deformed with operators of the first type is

$$W = h\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)\,, \tag{2.1}$$

where β is allowed to have an imaginary part. We will call this theory *β-deformed $\mathcal{N}=4$ SYM*. Our notation for the deformation and coupling parameters is related to the one in [34] as follows:

$$\beta \equiv \beta_d - i\kappa_d\,, \qquad \beta_d = \gamma - \tau_{1s}\sigma_d\,, \qquad \kappa_d = \tau_{2s}\sigma_d\,, \tag{2.2}$$

$$\tau_s = \tau_{1s} + i\tau_{2s} = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2_{\mathrm{YM}}}\,. \tag{2.3}$$

If β is real, i.e. $\sigma_d = 0$ we have

$$\beta_{\sigma_d=0} = \gamma\,. \tag{2.4}$$

We use the subscript d to indicate "deformation" and the subscript s to indicate "string coupling" (in an attempt to avoid confusion with the worldsheet coordinates τ and σ).

The deformation leading to (2.1) preserves the Cartan subalgebra of the SO(6) R-symmetry of $\mathcal{N}=4$ SYM and the diagonal U(1) is the R-symmetry of the manifest $\mathcal{N}=1$ supersymmetry algebra. In its general form (2.1) does not lead to a conformal field theory; requiring conformal invariance gives a constraint relating h, β and the gauge coupling constant. To two-loop order this constraint can be found using the general results about finiteness of $\mathcal{N}=1$ supersymmetric gauge theories [32] (we assume normalizations such that $h=1$ in $\mathcal{N}$=4 SYM theory)[6]

$$|h|^2(|e^{i\pi\beta}|^2 + |e^{-i\pi\beta}|^2) = 2\,. \tag{2.5}$$

Integrating out both the F and D auxiliary fields and restricting to the scalar sector, one finds the following scalar potential[7]

$$\begin{aligned} V \;=\; & |h^2\mathbf{q}^{-1}|\,\mathrm{Tr}\left[|\Phi_1\Phi_2 - \mathbf{q}\Phi_2\Phi_1|^2 + |\Phi_2\Phi_3 - \mathbf{q}\Phi_3\Phi_2|^2 + |\Phi_3\Phi_1 - \mathbf{q}\Phi_1\Phi_3|^2\right] + \\ & + \mathrm{Tr}\left[([\Phi_1,\,\bar\Phi^1] + [\Phi_2,\,\bar\Phi^2] + [\Phi_3,\,\bar\Phi^3])^2\right] \end{aligned} \tag{2.6}$$

$$\mathbf{q} \equiv e^{-2\pi i\beta}\,, \qquad q_d \equiv |\mathbf{q}| = e^{-2\pi\kappa_d}\,. \tag{2.7}$$

When $\mathbf{q}$ is a primitive n-th root of unity it was known for a while [42] that the dual geometry is that of the near-horizon D3-branes on orbifolds with discrete torsion. Until the recent work of [34] it was not clear, however, what is the gravity dual for the general *continuous* deformation parameter.

Below we will not review the derivation and results of [34] in detail concentrating instead on the world sheet implications of their construction.

[6] We are grateful to D. Freedman for pointing out an error in this condition in the original version of this paper.

[7] We use the same notation Φ_i for the first component of scalar superfields in (2.1).

2.2 Deformed type-IIB supergravity background and the corresponding superstring theory

The supergravity background dual to the β-deformed SYM theory was constructed in [34] using a combination of T-dualities and a shift on the isometries of the 5-sphere part of $AdS_5 \times S^5$. In addition, a non-trivial S-duality transformation was necessary for the construction of the background dual to the gauge theory with *complex* deformation parameter β, i.e. with $\sigma_d \neq 0$.

Let us begin with the simplest case of the real deformation β (i.e. when $\mathbf{q}$=pure phase, $q_d = 1$). Since in this case the construction of the supergravity background requires only the use of T-duality and a shift, we can implement it directly at the level of the world sheet string theory. The starting point should then be the classical Green-Schwarz action in $AdS_5 \times S^5$ [43].[8]

Through the usual implementation of T-duality as a 2-dimensional duality (with the well-known quantum dilaton shift properly taken into account [45]), or directly using the T-duality transformations spelled out in the context of the Green-Schwarz string in [46, 47], it is then not too hard to find the world sheet type-IIB superstring action resulting from applying the TsT transformation to the $AdS_5 \times S^5$ string action. The existence of such direct world-sheet construction of the "β-deformed" superstring action exposes three important facts:

1. Since the T-duality and the coordinate transformations should preserve the 2-dimensional conformal invariance of the string theory, there should exist a renormalization scheme in which the space-time background obtained in this way does not receive α' corrections even though it preserves only eight [34] supercharges (3/4 of supersymmetries are broken if β is not integer).

2. The 2-dimensional duality maps solutions of the original string equations of motion into solutions of the dual equations of motion; this construction then represents a formal way of generating solutions of the classical equations of motion for a string moving in the dual string background. One should still take care of the global issues like periodicity of world sheet coordinates by appropriate twists which implies that the original and the TsT transformed string theories are not actually equivalent for non-integer β.

3. The construction of the one (spectral) parameter family of flat currents or the Lax pair which demonstrates the classical integrability of the free string theory in $AdS_5 \times S^5$ relies only on the classical string equations of motion [48 − 51]. This implies that at least part (in fact, an infinite subset) of the integrals of motion of $AdS_5 \times S^5$ sigma

[8] For the purpose of eventually considering a quantum string theory the starting point may be the light-cone κ-symmetry gauge fixed action of [44]; this action also has a desirable feature that the isometries necessary to perform the T-duality transformations [34] are not obscured by the gauge fixing condition.

model are mapped into integrals of motion of the deformed sigma model, suggesting that in the case of real β the deformed string theory is also classically integrable.[9]

The resulting supergravity background for string theory dual of the real β-deformation of the $\mathcal{N}=4$ SYM is [34]:

$$
\begin{aligned}
ds^2_{\rm str} &= R^2\left[ds^2_{{}_{AdS_5}} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + \tilde\gamma^2 G\rho_1^2\rho_2^2\rho_3^2(\sum_{i=1}^{3} d\phi_i)^2\right] && (2.8)\\
B_2 &= R^2\tilde\gamma G w_2\,, \qquad w_2 \equiv \rho_1^2\rho_2^2 d\phi_1 d\phi_2 + \rho_2^2\rho_3^2 d\phi_2 d\phi_3 + \rho_3^2\rho_1^2 d\phi_3 d\phi_1\,, && (2.9)\\
e^{\phi} &= e^{\phi_0}G^{1/2}\,, \qquad \chi = 0\,, && (2.10)\\
G^{-1} &\equiv 1+\tilde\gamma^2 Q\,, \qquad Q \equiv \rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_1^2\rho_3^2\,, \qquad \sum_{i=1}^{3}\rho_i^2 = 1\,, && (2.11)\\
C_2 &= -12R^2 e^{-\phi_0}\tilde\gamma w_1 d\psi\,, \qquad dw_1 \equiv \cos\alpha\sin^3\alpha\sin\theta\cos\theta d\alpha d\theta\,, && (2.12)\\
F_5 &= 4R^4 e^{-\phi_0}(\omega_{{}_{AdS_5}} + G\omega_{{}_{S^5}})\,, \qquad \omega_{{}_{S^5}} \equiv dw_1 d\phi_1 d\phi_2 d\phi_3\,. && (2.13)
\end{aligned}
$$

Here B_2 is the NS-NS 2-form potential, ϕ is the dilaton and $d\chi$, dC_2 and F_5 are the R-R field strengths. The angles ψ and θ,α appearing in dw_1 are defined as follows

$$
\psi = \frac{1}{3}(\phi_1+\phi_2+\phi_3)\,, \qquad \rho_1 = \sin\alpha\cos\theta\,, \qquad \rho_2 = \sin\alpha\sin\theta\,, \qquad \rho_3 = \cos\alpha\,. \quad (2.14)
$$

The standard $AdS_5\times S^5$ background is recovered after setting the deformation parameter $\tilde\gamma = R^2\gamma \sim \beta$ to zero (then $G=1$, $\phi=\phi_0$ and $B_2 = C_2 = 0$). The parameter $\tilde\gamma$ was denoted as $\hat\gamma$ in [34] and other parameters are

$$
\begin{aligned}
g_s &= e^{\phi_0} = \frac{1}{\tau_{2s}} = \frac{g^2_{\rm YM}}{4\pi}\,, && (2.15)\\
R^4 &= e^{\phi_0}R_E^4 = 4\pi g_s N = N g^2_{\rm YM} \equiv \lambda\,, \qquad \alpha' = 1\,, && (2.16)\\
\tilde\gamma &= \gamma R^2\,. && (2.17)
\end{aligned}
$$

In order for the superstring theory defined on this background to have a consistent perturbative expansion, the parameter $\tilde\gamma$ (cf. (2.3)) and thus the string-theory action should be independent of the string coupling constant e^{ϕ_0}. Indeed, observing that the superstring action depends on the RR field strengths $F_3 = dC_2$ and F_5 in combination with e^{ϕ} factor [52] and also that the vielbein components of F_3 and F_5 both scale as R we conclude that the inverse string tension expansion of the classical ($g_s = 0$) superstring action in the above background will depend only on $1/R^2 = \sqrt{\lambda}$ and the parameter $\tilde\gamma$ but not on $e^{-\phi_0} = \tau_{2s}$. This is what is needed to be able to use a systematic semiclassical approximation to string theory, with $\tilde\gamma$ playing the role of a regular deformation parameter of the supergravity background and of the superstring action.

[9] After all, a non-technical meaning of integrability of a (2-d) field theory is that there exists at least an implicit but systematic way of classifying all classical solutions. The usual necessary condition for that in a field-theory context where there is an infinite number of degrees of freedom is the existence of an infinite number of conserved charges. This practical criterion is certainly satisfied for this deformed string theory.

Next, let us follow [34] and apply S-duality[10] before and after the TsT-transformation leading to the above type IIB supergravity background.[11] This step may be viewed as a formal trick to generate a type-IIB supergravity solution dual to 2-parameter (complex β) deformation of the $\mathcal{N}=4$ SYM theory.[12] We finish with the following expressions (equivalent to the ones in [34] but written for the string-frame metric and using the notation introduced above):

$$ds^2_{\rm str} = R^2 H^{1/2}\left[ds^2_{AdS_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + (\tilde\gamma^2+\tilde\sigma^2)G\rho_1^2\rho_2^2\rho_3^2(\sum_{i=1}^{3} d\phi_i)^2\right] \quad (2.18)$$

$$B_2 = R^2\left(\tilde\gamma G w_2 - 12\tilde\sigma w_1 d\psi\right), \quad (2.19)$$

$$e^{\phi} = e^{\phi_0}G^{1/2}H\,, \qquad \chi = \chi_0 + e^{-\phi_0}\tilde\sigma\tilde\gamma H^{-1}Q\,, \quad (2.20)$$

$$G^{-1} \equiv 1 + (\tilde\gamma^2+\tilde\sigma^2)Q\,, \qquad H \equiv 1 + \tilde\sigma^2 Q\,, \quad (2.21)$$

$$C_2 = R^2[(\chi_0\tilde\gamma - e^{-\phi_0}\tilde\sigma)Gw_2 - 12(e^{-\phi_0}\tilde\gamma + \chi_0\tilde\sigma)w_1 d\psi]\,, \quad (2.22)$$

$$F_5 = 4R^4 e^{-\phi_0}(\omega_{AdS_5} + G\omega_{S^5})\,. \quad (2.23)$$

Here $\tilde\sigma$ is the new deformation parameter and we have used the following definitions (cf. [34] and (2.3)):

$$\chi_0 + ie^{-\phi_0} = \tau_s = \tau_{1s} + i\tau_{2s} = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2_{\rm YM}}\,, \quad (2.24)$$

$$\tilde\beta = \tilde\gamma - i\tilde\sigma = \beta R^2 = (\gamma - \tau_s\sigma_d)R^2\,, \quad (2.25)$$

$$\tilde\gamma = (\gamma - \tau_{1s}\sigma_d)R^2 = \beta_d R^2\,, \qquad \tilde\sigma = \tau_{2s}\sigma_d R^2 = \kappa_d R^2\,. \quad (2.26)$$

We shall assume that the parameter $\tilde\beta$ is independent of the string coupling constant.

The classical superstring action in this S-dual background is straightforward to write down, at least to quadratic order in fermions.[13] Here, however, we have no good reason to believe that the S-dual background invariant only under eight supercharges will not be deformed by α'/R^2 corrections: we may then need to modify the classical superstring action by extra α'/R^2 correction terms in order to ensure its quantum 2d conformal invariance. These extra terms may depend also on background field strengths and thus on other parameters like ϕ_0 and χ_0. For simplicity, let us assume that this does not happen. Then the "NS-NS" part of the superstring action will obviously depend only on R^2 and $\tilde\beta$ (more precisely, on $\tilde\gamma$ and $\tilde\sigma$), i.e. will have no dependence on the string coupling constant τ_s. As in the $\sigma_d = 0$ case discussed above, this is important in order to be able to develop a semiclassical expansion for the energies of the corresponding string states which we will be interested in.

[10] The required S-duality transformation depends on one real parameter, σ, and acts on the complex coupling constant $\tau = \chi_0 + ie^{-\phi_0}$, thus introducing the two new real parameters χ_0 and σ.

[11] It is important to note that since there are two S-duality transformations, the initial and final backgrounds are at weak string coupling.

[12] It may still be of interest to study the interplay between the corresponding fundamental string and D-string actions that probe, respectively, the S-dual backgrounds. In general, starting with a D-string in a given background and "rotating away" the 2d gauge field by a 2d duality transformation one ends up with the fundamental string action coupled to the S-dual background [53].

[13] Note that here the string metric is no longer a direct product of AdS_5 and S^5_β but rather conformal to it. That means that in general the classical string motions in AdS_5 and S^5_β parts will no longer factorize.

One may still worry (legitimately) about the $e^{-\phi_0}$ and χ_0 dependence of the fermionic couplings to the RR field strengths. The fundamental string action is certainly not invariant under the S-duality applied to a background in which it is moving. However, to quadratic order in fermions (which is enough to study 1-loop correction to semiclassical approximation) one is able to argue that this dependence is fake and just reflects the construction via S-duality transformation: the superstring fermions couple though the generalized covariant derivative to a particular combinations of fluxes which is covariant under the S-duality. Indeed, this generalized covariant derivative is the same as the one appearing in the type-IIB supergravity variation of the gravitino field, and the latter (like the full type-IIB supergravity action) is invariant under S-duality. Explicitly, the RR 3-form field strength dC_2 enters the quadratic fermionic action though the combination $e^{\phi}\,(dC_2 - \chi dB_2)$, and computing this combination on the above background we observe that the dependence on the dilaton and the RR scalar constants e^{ϕ_0} and χ_0 indeed cancels out. The spinor covariant derivative contains also the RR scalar field strength term $e^{\phi}d\chi$, and it is again independent of ϕ_0 and χ_0.

We conclude that, at least *to quadratic order in fermions*, the corresponding superstring action does depend only on the overall factor of string tension R^2 and on the parameters $\tilde{\gamma}$ and $\tilde{\sigma}$ but not on the string coupling τ_s. The general case of the full string action is, however, unclear, i.e. the perturbative semiclassical study of the gauge-string duality in case of complex β deformation appears to be problematic.

3. Semiclassical string expansion: limits and parameters

In the semiclassical approach to string spectrum one starts with a classical string action on a 2d cylinder and computes the energies of classical string solutions. One may then compute also quantum (α'/R^2) string sigma model corrections, for which one will need to include also the fermionic terms in the string action (and, in general, to take into account the dilaton coupling, cf. [76]). In the case of multispin string states considered in [8] the quantum string corrections were suppressed in the large spin limit ($J \to \infty$), and the same is expected to happen in the present "deformed" case.

The relevant bosonic part of the classical string action depends on the metric G_{MN} and the NS-NS 2-form field B_{MN}, i.e. is given by

$$S_B = -\frac{R^2}{2}\int d\tau \int \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N\, G_{MN} - \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N\, B_{MN}\right], \qquad (3.1)$$

where we set $\alpha' = 1$, $\epsilon^{01} = 1$, and $\gamma^{\alpha\beta} \equiv \sqrt{-h}\, h^{\alpha\beta}$ ($h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature, i.e. in the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1,1)$). The string-frame metric and the B_2-field are given by (2.8) and (2.9) or, for complex β, by (2.18) and (2.19).

The parameters in the string sigma model (3.1) are then the AdS_5 radius $R^2 = \sqrt{\lambda}$ (playing the role of the coupling constant of the sigma model) and the complex deformation parameter $\tilde{\beta}$ or, equivalently, the two real parameters $\tilde{\gamma}$ and $\tilde{\sigma}$. Note that since the metric and B_2-field have regular expansion in powers of $\tilde{\gamma}$ and $\tilde{\sigma}$, the same is true for the classical string action, with the zero-order term being the $AdS_5 \times S^5$ bosonic sigma model with the S^5 part being $\int \sum_{i=1}^{3}[(\partial\rho_i)^2 + \rho_i^2(\partial\phi_i)^2]$.

The semiclassical approximation [5, 6] is useful for states that carry large energy and which can thus be approximated by classical string solutions with $E = \sqrt{\lambda}\mathcal{E} + \cdots \gg 1$. Here $\mathcal{E}$ (depending on parameters of classical solution) is fixed in the semiclassical expansion, and quantum corrections are small provided the sigma model coupling constant $\sqrt{\lambda}$ is large. In present AdS/CFT context, the energy of the corresponding quantum string states will in general involve functions of λ interpolating between the "perturbative string theory" ($E = \sqrt{\lambda}(\mathcal{E} + \frac{b_1}{\sqrt{\lambda}} + \cdots)$) and the "perturbative gauge theory" ($E = c_1\lambda + c_2\lambda^2 + \cdots$) regimes. However, for a special (large) class of semiclassical $AdS_5 \times S^5$ string states such interpolation functions happen to be absent in the first few leading terms of the expansion of the energy in large quantum numbers [8].

Here we would like to study the analog of this class of multi-spin states in the β-deformed theory. The string solutions of the type we are going to consider below[14] have large total angular momentum $J = \sqrt{\lambda}\mathcal{J}$ in (deformed) S^5 directions and their energy admits a regular expansion in powers of the effective coupling constant

$$\tilde{\lambda} = \frac{\lambda}{J^2} = \frac{1}{\mathcal{J}^2} \tag{3.2}$$

which, in the semiclassical expansion, is assumed to be fixed and independent of λ. Since the field theory perturbative expansion for anomalous dimensions of the corresponding operators happens to have the same dependence on both λ *and* J, one may then attempt to compare the string and field theory expressions.

The string solutions in the β-deformed background will depend on the deformation parameters $\tilde{\gamma}$ and $\tilde{\sigma}$. Since the deformation is smooth, the classical energy of a multi-spin solution should then have the following expansion[15]

$$E = \sqrt{\lambda}\,\mathcal{E}(\tilde{\beta}, \mathcal{J}) = \sqrt{\lambda}\left[\mathcal{E}_0(\mathcal{J}) + \tilde{\beta}\, f_1(\mathcal{J}) + \tilde{\beta}^2\, f_2(\mathcal{J}) + \tilde{\beta}^3\, f_3(\mathcal{J}) + \cdots\right], \tag{3.3}$$

where $\mathcal{E}_0(\mathcal{J})$ represents the energy in the $AdS_5 \times S^5$ case, i.e. it has regular large $\mathcal{J}$ or small $\tilde{\lambda}$ expansion [8]

$$E_0 = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) = J f_0(\tilde{\lambda}) = J\left(1 + c_1\tilde{\lambda} + c_2\tilde{\lambda}^2 + \cdots\right). \tag{3.4}$$

To be able to compare to gauge-theory perturbative expansion, we would like to find out when the $\tilde{\beta}$-dependent terms in (3.3) admit a similar expansion. To this end we notice that $\tilde{\beta} \equiv R^2\beta$ in (2.26) can be written in the form

$$\tilde{\beta} = \frac{R^2}{J}\beta J = \frac{\sqrt{\lambda}}{J}\,\beta J = \sqrt{\tilde{\lambda}}\,\beta J\,, \tag{3.5}$$

[14] More generally, the relevant string configurations are "fast strings" [54] that may not only rotate but also pulsate [55], etc.

[15] The string energy is real and depends separately on $\tilde{\gamma}$ and $\tilde{\sigma}$, and the dependence on each of the two parameters is regular since the background fields in (2.18), (2.19) admit regular expansions in $\tilde{\gamma}, \tilde{\sigma}$. To simplify the expressions, instead of writing E as a series in $\tilde{\gamma}$ and $\tilde{\sigma}$ we shall write it symbolically as a series in $\tilde{\beta}$.

where β is the deformation parameter on the gauge theory side. Being a parameter of the classical string action $\tilde{\beta}$ is fixed in the semiclassical expansion in which $\tilde{\lambda}$ is also fixed; it then follows that βJ should also be kept fixed in the limit $J \to \infty$. Equivalently, we need to assume that the string-theory perturbative expansion corresponds to the limit when $J \to \infty,\ \ \lambda = \tilde{\lambda} J^2 \to \infty$ and $\beta = \frac{\tilde{\beta}}{\sqrt{\tilde{\lambda}} J} \to 0$.

We will take into account that $\beta \sim \frac{1}{J}$ corresponds to the "semiclassical string" scaling limit in the subsequent sections when we will (i) define the relevant continuum limit of the spin chain hamiltonian which represents the one-loop dilatation operator in the "$su(2)_\beta$" subsector of the β-deformed gauge theory, and (ii) when we will consider the thermodynamic limit of the Bethe ansatz for the spin chain. This will amount to isolating a class of "macroscopic" spin-wave states or gauge theory operators which are dual to the semiclassical string states.

Using the relation (3.5), we can rewrite the expression for the classical energy (3.3) as follows

$$E = J\left[f_0(\tilde{\lambda}) + \tilde{\lambda}\beta J\, f_1(\mathcal{J}) + \tilde{\lambda}(\beta J)^2\, \frac{f_2(\mathcal{J})}{\mathcal{J}} + \tilde{\lambda}^2(\beta J)^3\, f_3(\mathcal{J}) + \cdots\right]. \tag{3.6}$$

In general, the coefficient functions $f_1, f_2, \ldots$ may have an expansion in powers of $\sqrt{\tilde{\lambda}}$, but in order for a semiclassical string energy to have a perturbative gauge theory interpretation it is necessary that (3.6) has a regular expansion in powers of $\tilde{\lambda}$. If this is the case, E may be formally rewritten as a series of terms containing powers of λ, β and $\frac{1}{J}$, i.e. in the same form which one expects to find for a perturbative anomalous dimension on the gauge-theory side. This boils down to the condition that $f_1, \mathcal{J}^{-1} f_2, f_3, \ldots$ should have regular expansions in powers of $1/\mathcal{J}^2 = \tilde{\lambda}$. Assuming this one finds

$$E = J\Big[1 + \tilde{\lambda}\left(c_1 + d_1\, \beta J + d_2\, (\beta J)^2\right) + O(\tilde{\lambda}^2)\Big]. \tag{3.7}$$

As a consequence, we observe that comparing to the gauge-theory 1-loop anomalous dimensions we will be able to probe only the terms of order $(\beta J)^k$ with $k \leq 2$.

To be able to compare the terms of higher orders in βJ one should take into account the world-sheet quantum corrections. Computing quantum string corrections does not pose a problem in the case of real β or $\sigma_d = 0$, but this is by far less clear in the general case of $\sigma_d \neq 0$. As we have pointed out in the previous section, in the case of an arbitrary complex deformation the RR fluxes depend explicitly on the string coupling τ_s, but this dependence drops out of the action at the quadratic fermionic level; then there should be no "anomalous" string coupling dependence at least at the 1-loop string sigma model level. Then the semiclassical loop expansion for the string energy should look like:

$$E = \sqrt{\lambda}\mathcal{E}_0(\tilde{\beta}, \mathcal{J}) + \mathcal{E}_1(\tilde{\beta}, \mathcal{J}) + \cdots. \tag{3.8}$$

In particular, including the world sheet one-loop quantum correction allows one to compare with gauge-theory 1-loop terms the terms in E that contain factors of $(J\beta)^3$ with $k \leq 3$.

4. Strings on $R_t \times S^3_\beta$ and large spin limit of the string sigma model

In this section we shall consider classical strings located at the center of AdS_5 and moving in the "S^3_β" part of the deformed 5-space in (2.18) defined by the condition (see (2.14))

$$\alpha = \frac{\pi}{2}\,, \qquad \text{i.e.} \qquad \rho_1 = \cos\theta\,, \qquad \rho_2 = \sin\theta\,, \qquad \rho_3 = 0\,, \tag{4.1}$$

which, as it is easy to see, is indeed a consistent truncation of the string equations of motion.[16] The resulting bosonic string action (3.1) is

$$\begin{aligned} S = -\frac{1}{2}R^2 \int d\tau \int \frac{d\sigma}{2\pi} \Big[\gamma^{\alpha\beta} H^{1/2} (-\partial_\alpha t \partial_\beta t + \partial_\alpha\theta\partial_\beta\theta + \\ + G\,\cos^2\theta\,\partial_\alpha\phi_1\partial_\beta\phi_1 + G\,\sin^2\theta\,\partial_\alpha\phi_2\partial_\beta\phi_2) - \\ - 2\epsilon^{\alpha\beta}\tilde\gamma\, G\,\sin^2\theta\,\cos^2\theta\,\partial_\alpha\phi_1\partial_\beta\phi_2\Big]\,, \end{aligned} \tag{4.2}$$

where

$$G = \left[1 + \frac{1}{4}(\tilde\gamma^2 + \tilde\sigma^2)\sin^2 2\theta\right]^{-1}, \qquad H = 1 + \frac{1}{4}\tilde\sigma^2 \sin^2 2\theta\,. \tag{4.3}$$

and we recall that the string theory deformation parameters $\tilde\gamma$ and $\tilde\sigma$ are related to the gauge theory deformation parameters γ and σ_d by the equations (2.26) and (2.3). To get an idea of how the β-deformation may influence the form of string solutions we shall first discuss a particular example which is a deformation of the simplest 2-spin circular string solution of [8, 12].

We shall then consider, following what was done in $\beta = 0$ case in [16, 40], the string sigma model on $R_t \times S^3_\beta$ in a particular gauge where $t \sim \tau$ and the density of the angular momentum in the two angular directions ϕ_1 and ϕ_2 is constant. This uniform gauge allows one to develop a systematic large spin or "fast-string" expansion, with the leading term in the action for "slow" variables having non-relativistic Landau-Lifshitz form. The solutions of this reduced sigma model are expected to be dual to operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2} + \cdots)$ from the holomorphic 2-scalar field sector of the gauge theory. Indeed, in the next section we shall show that exactly as in the undeformed case [15, 16], this string action is the same as the "Landau-Lifshitz" action describing macroscopic spin waves of the spin chain hamiltonian representing the gauge theory dilatation operator in the 2-scalar sector.

4.1 A circular 2-spin solution

Let us first discuss the case of $\sigma_d = 0$ where the deformed background can be obtained from $AdS_5 \times S^5$ by means of a T-duality on one circle of S^5, a shift of another angle variable, followed by another T-duality. Let us use the same circular string ansatz as the one in [12]

$$\phi_1 = \mathrm{w}_1\tau + m_1\sigma\,, \qquad \phi_2 = \mathrm{w}_2\tau + m_2\sigma\,, \qquad \rho_1 = \cos\theta_0\,, \qquad \rho_2 = \sin\theta_0\,, \tag{4.4}$$

with constant θ_0, integer m_1, m_2 and and $t = \kappa\tau$. Then, it is not difficult to check that if

$$m_1^2 - \mathrm{w}_1^2 = m_2^2 - \mathrm{w}_2^2\,, \qquad \theta_0 = \frac{\pi}{4} \tag{4.5}$$

[16] More precisely, in the case of $\tilde\sigma \neq 0$ the space where string moves is not a direct product $R_t \times S^3_\beta$ but is conformal to it.

we get a solution to the string equations of motion following from (3.1). Furthermore, assuming that both angular momenta $\mathcal{J}_1$ and $\mathcal{J}_2$ corresponding to the angles $\phi_{1,2}$ are positive, the Virasoro constraints are satisfied if

$$\begin{aligned} \mathcal{J}_1 &= \mathcal{J}_2 \equiv \frac{1}{2}\mathcal{J}\,, \qquad m_1 = -m_2 \equiv m\,, \\ \mathrm{w}_1 &= \mathrm{w}_2 = \mathcal{J} + \frac{1}{2}\tilde{\gamma}\Big(m + \frac{1}{2}\tilde{\gamma}\mathcal{J}\Big)\,. \end{aligned} \tag{4.6}$$

This solution has a smooth limit as $\tilde{\gamma} = \gamma R^2 \to 0$. The corresponding energy is:

$$E = R^2\sqrt{\mathcal{J}^2 + \Big(m + \frac{1}{2}\tilde{\gamma}\mathcal{J}\Big)^2} = \sqrt{J^2 + \lambda\Big(m + \frac{1}{2}\gamma J\Big)^2}\,. \tag{4.7}$$

The undeformed case corresponds to $\tilde{\gamma} = 0$, i.e. we conclude that the deformation amounts simply to a shift of the string winding number n. Since $m_1 = -m_2$, then $\phi_1(2\pi) - \phi_2(2\pi) - (\phi_1(0) - \phi_2(0)) = \gamma J$ which implies that the role of the phase (γ) part of the deformation is to twist the boundary conditions of the angle $\phi_1 - \phi_2$ of S^3 in the σ direction by $\gamma J = \tilde{\gamma}\mathcal{J}$.

Let us note also that the expansion of the energy (4.7)

$$E = J\left[1 + \frac{1}{2}\tilde{\lambda}\left(m + \frac{1}{2}\gamma J\right)^2 + \cdots\right] \tag{4.8}$$

provides an example of the expected regular expansion (3.7); in particular, at the "1-loop" (order λ) level we get only the terms of order γJ and $(\gamma J)^2$.

It is easy to generalize the above solution to the case of the background with $\sigma_d \neq 0$. We choose the same circular string ansatz (4.4) with constant θ_0, and get the same relations (4.5). The Virasoro constraints are satisfied if again $\mathcal{J}_1 = \mathcal{J}_2$, $m_1 = -m_2$, and

$$\mathrm{w}_1 = \mathrm{w}_2 = \frac{\mathcal{J} + \tilde{\gamma}(m + \frac{1}{2}\tilde{\gamma}\mathcal{J}) + \frac{1}{2}\tilde{\sigma}^2\mathcal{J}}{\sqrt{1 + \frac{1}{4}\tilde{\sigma}^2}}\,. \tag{4.9}$$

This solution also has a smooth limit as $\tilde{\beta} = \tilde{\gamma} - i\tilde{\sigma} \to 0$. The string energy now is:

$$E = R^2\sqrt{\mathcal{J}^2 + \Big(m + \frac{1}{2}\tilde{\gamma}\mathcal{J}\Big)^2 + \frac{1}{4}\tilde{\sigma}^2\mathcal{J}^2}\,. \tag{4.10}$$

We see again that the phase of the deformation $\mathbf{q}$ twists the boundary condition of $\phi_1 - \phi_2$ in the σ direction by $\tilde{\gamma}\mathcal{J}$; and we will see below that the $\tilde{\sigma}^2$ term comes from a potential term in the reduced ("Landau-Lifshitz") sigma model action. This expression is also consistent with (3.7). The energy can be also rewritten as

$$E = R^2\sqrt{\mathcal{J}^2 + \Big|m + \frac{1}{2}\tilde{\beta}\mathcal{J}\Big|^2} = J\sqrt{1 + \tilde{\lambda}\Big|m + \frac{1}{2}\beta J\Big|^2}\,. \tag{4.11}$$

This expression for the string energy suggests connection with the SL(2) duality transformation used to obtain the deformed background.

4.2 "Fast string" expansion of the sigma model action

Let us now study the structure of the string sigma model action in the "2-spin" sector in the limit of large total spin. The discussion follows closely the one in the undeformed case [15, 16, 40]. Setting

$$\phi_1 = \zeta + \eta\,, \qquad \phi_2 = \zeta - \eta\,, \qquad \rho_1 = \cos\theta\,, \qquad \rho_2 = \sin\theta\,, \qquad \rho_3 = 0\,, \tag{4.12}$$

we may treat ζ (which is the analog of the angle ψ in the 2-spin case and whose conjugate momentum is the total spin $J = J_1 + J_2$) as a "fast" angular variable while η and θ will be "slow" variables whose time evolution will be suppressed. Then the relevant part of the bosonic sigma model lagrangian (3.1) in the background (2.18), (2.19) can be written as (see (4.2), (4.3))

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}R^2\sqrt{-h}h^{\alpha\beta}H^{1/2}\Big[& -\partial_\alpha t\partial_\beta t + \partial_\alpha\theta\partial_\beta\theta + G(\partial_\alpha\zeta\partial_\beta\zeta + \\ & + \partial_\alpha\eta\partial_\beta\eta + 2\cos 2\theta\,\partial_\alpha\zeta\partial_\beta\eta)\Big] - \\ -\frac{1}{2}R^2\tilde{\gamma}\,G\sin^2 2\theta\,\epsilon^{\alpha\beta}\partial_\alpha\zeta\partial_\beta\eta\,. \end{aligned}\tag{4.13}$$

To implement the uniform gauge fixing one may either consider the phase-space action and fix p_ζ =const or, equivalently, first do T-duality (i.e. 2-d duality) in ζ direction in the above lagrangian and gauge-fix $\tilde{\zeta} = J\sigma$. The lagrangian after T-duality takes the form

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\sqrt{-h}h^{\alpha\beta}R^2H^{-1/2}\Big[& \frac{\partial_\alpha\tilde{\zeta}\partial_\beta\tilde{\zeta}}{R^4G} + \frac{\tilde{\gamma}}{R^2}\sin^2 2\theta\,\partial_\alpha\tilde{\zeta}\partial_\beta\eta + \\ & + H(-\partial_\alpha t\partial_\beta t + \partial_\alpha\theta\partial_\beta\theta) + \sin^2 2\theta\,\partial_\alpha\eta\partial_\beta\eta\Big] \\ -\cos 2\theta\,\epsilon^{\alpha\beta}\,\partial_\alpha\tilde{\zeta}\partial_\beta\eta\,. \end{aligned}\tag{4.14}$$

Note that the resulting "Wess-Zumino" term is identical to the one in the absence of the deformation.

Imposing now the gauge (here τ and σ are world-sheet coordinates)

$$t = \tau\,, \qquad \tilde{\zeta} = J\sigma\,, \qquad J = R^2\mathcal{J} = \sqrt{\lambda}\mathcal{J}\,, \tag{4.15}$$

and solving for the world sheet metric, we find:

$$\mathcal{L} = J\cos 2\theta\;\dot{\eta} - \sqrt{-\det h}\,, \tag{4.16}$$

where

$$\begin{aligned} -\det h = R^4\Bigg[& \left(H^{-1}\sin^2 2\theta\,\dot{\eta}\left(\eta' + \frac{1}{2}\tilde{\gamma}\mathcal{J}\right) + \dot{\theta}\theta'\right)^2 + \\ & + \left(1 - \dot{\theta}^{\,2} - H^{-1}\sin^2 2\theta\,\dot{\eta}^2\right)\left(\mathcal{J}^2 + H\,\theta'^2 + \sin^2 2\theta\,|\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right)\Bigg]\,, \end{aligned}\tag{4.17}$$

where

$$\left|\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}\right|^2 = \left(\eta' + \frac{1}{2}\tilde{\gamma}\mathcal{J}\right)^2 + \frac{1}{4}(\tilde{\sigma}\mathcal{J})^2\,. \tag{4.18}$$

The circular solution we discussed in the previous subsection (4.4), (4.9) corresponds to the static solution of this action:

$$\theta = \frac{\pi}{4}\,, \qquad \eta = m\sigma\,, \qquad \sqrt{-\det h} = J\sqrt{1+\tilde{\lambda}|m+\frac{1}{2}\tilde{\beta}\mathcal{J}|^2}\,, \tag{4.19}$$

and so the hamiltonian part of (4.16) reproduces the energy of this solution given in (4.11).

To isolate the sector of "fast strings" with regular expansion of the energy in $\tilde{\lambda}$ we should take J to be large, i.e. $\tilde{\lambda} = \frac{\lambda}{J^2} \to 0$, and expand in time derivatives (which, when eliminated using leading-order equations of motion [16], will contribute to higher orders in $\tilde{\lambda}$). Then

$$\sqrt{-\det h} = J\sqrt{1+\tilde{\lambda}\left[(1+\frac{1}{4}\tilde{\sigma}^2\sin^2 2\theta)\ \theta'^2 + \sin^2 2\theta\,|\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right]} + O(\dot{\eta},\dot{\theta})\,. \tag{4.20}$$

The $\mathcal{J}\to\infty$ expansion or expansion in powers of $\tilde{\lambda}$ makes sense assuming the parameter $\tilde{\beta}\mathcal{J}$ is fixed in this limit, i.e. $\tilde{\sigma}\sim\mathcal{J}^{-1}$, $\tilde{\beta}\sim\mathcal{J}^{-1}$. Then to the leading order in $\tilde{\lambda}$ the prefactor of the θ'^2 term contributes simply 1 and we get for the resulting lagrangian

$$\mathcal{L} = J\left[\cos 2\theta\ \dot{\eta} - \frac{1}{2}\tilde{\lambda}\left(\theta'^2 + \sin^2 2\theta\,|\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right) + O(\tilde{\lambda}^2)\right]. \tag{4.21}$$

Thus the real part of the deformation parameter β twists the boundary conditions of the world-sheet field η while the imaginary part adds a nontrivial potential term (cf. (4.18)).[17]

This action leads to the anisotropic Landau-Lifshitz equations with twisted boundary conditions for the η field (the anisotropy potential term is present only if $\tilde{\sigma}\neq 0$). These equations are known to be integrable [56], and this suggests that the string theory in this 2-spin sector is integrable (at least to the leading order in the $\tilde{\lambda}$ expansion).

The action (4.21) admits a simple classical solution with two unequal angular momenta J_1, J_2 that generalizes the solution in (4.19) and should be a large $\mathcal{J}$ limit of a $J_1\neq J_2$ generalization of the solution (4.6) (i.e. the counterpart of the circular solution of [12]). Translating the string ansatz (4.4) into the variables (4.12) appearing in (4.21) we find

$$\theta = \theta_0\,, \qquad \eta \equiv \frac{1}{2}(\phi_1-\phi_2) = w\tau + \frac{1}{2}\mathrm{m}\sigma\,, \tag{4.22}$$

where $\mathrm{m} = m_1 - m_2$ is an integer and w is a constant which is to be determined by solving equations following from (4.21).[18] It is easy to see that the constant θ assumption implies that η equation of motion is trivialy satisfied while the θ equation fixes w in terms of m

[17] Let us note that even though the lagrangian (4.21) formally depends on the real and imaginary parts of the deformation parameter β through $\tilde{\gamma} - i\tilde{\sigma}$, it is not clear that the same is true for energies of all semiclassical solutions (as we have seen in the previous subsection, for some of them this is still the case, cf. (4.11)). Indeed, the equations of motion following from (4.21) depend separately on $\tilde{\gamma}$ and $\tilde{\sigma}$ and this may continue once their solutions are plugged into the hamiltonian.

[18] It will be the leading-order term in the large $\mathcal{J}$ expansion of $\frac{1}{2}(w_1-w_2)$. Note that the condition $m_1J_1+m_2J_2 = 0$ following from the Virasoro constraint in the string theory setting, reappears in the Landau-Lifshitz context from the momentum quatization condition $\frac{1}{2}(m_1+m_2) = \int\frac{d\sigma}{2\pi}\partial_\sigma\zeta = -\int\frac{d\sigma}{2\pi}\cos 2\theta\ \partial_\sigma\eta$.

and θ_0. The latter is determined in terms of the angular momenta. Indeed, the relation between η and ϕ_1 and ϕ_2 (4.22) implies that the momentum conjugate to η is given by the difference between the conjugate momenta of ϕ_1 and ϕ_2, i.e. $J_1 - J_2$. Thus,

$$J_1 - J_2 = J \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos 2\theta_0 = J \cos 2\theta_0 \quad \longrightarrow \quad \sin^2 2\theta_0 = 4\alpha(1-\alpha) \qquad \text{with} \qquad \alpha \equiv \frac{J_2}{J} \,. \tag{4.23}$$

Then we find that

$$w = -\frac{1}{4}\tilde{\lambda}(1-2\alpha)\Big|\mathrm{m} + \tilde{\beta}\mathcal{J}\Big|^2 \,. \tag{4.24}$$

The equation (4.23) also implies that, to leading order in the $\tilde{\lambda}$ expansion, the energy of the solution (4.22) as determined from (4.21) is given by

$$E_0 = \frac{1}{8} J\tilde{\lambda} \sin^2 2\theta_0 \, \Big|\mathrm{m} + \tilde{\beta}\mathcal{J}\Big|^2 = \frac{\lambda}{2J}\alpha(1-\alpha)\Big|\mathrm{m} + \tilde{\beta}\mathcal{J}\Big|^2 \,. \tag{4.25}$$

Note that in the case of $J_1 = J_2$ ($\alpha = 1/2$) in (4.19) we have $\mathrm{m} = 2m$ and (4.25) agrees with the leading term in the expansion of the square root in (4.19).

We conclude that, as in the undeformed case, the deformed ("twisted" and "anisotropic") Landau-Lishitz action (4.21) has a simple rational solution for generic value of $J_1 - J_2$. We will reproduce (4.25) in section 8.2 from the Bethe ansatz for the corresponding spin chain.

One can also compute the first subleading correction to the 2d hamiltonian corresponding to (4.16)

$$\mathrm{H} = J \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H} \,, \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots \,, \tag{4.26}$$

$$\mathcal{H}_0 = 1 \,, \qquad \mathcal{H}_1 = \frac{1}{2}\tilde{\lambda}\left(\theta'^2 + \sin^2 2\theta \, |\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right) , \tag{4.27}$$

using the same procedure of eliminating time derivatives with the help of the leading-order equations of motion as in [16]. One finds then for the λ^2 term:

$$\begin{aligned} \mathcal{H}_2 = -\frac{1}{8}\tilde{\lambda}^2 \Bigg[& \left(\theta'^{\,2} + \sin^2 2\theta \, |\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right)^2 - (\tilde{\sigma}\mathcal{J})^2 \sin^2 2\theta \, \theta'^{\,2} \\ & + \left[4\cos 2\theta \, \theta'(\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}) + \sin 2\theta \, \eta''\right]^2 + \left(\theta'' - \sin 4\theta \, |\eta' + \frac{1}{2}\tilde{\beta}\mathcal{J}|^2\right)^2 \Bigg] . \end{aligned} \tag{4.28}$$

Note that this correction to the Landau-Lifshitz action depends separately on the real and imaginary parts of the deformation parameter β. This can be traced to the fact that while in the leading order action the deformation parameter enters only as a (complex) twist in the boundary conditions of the η coordinate, this is not true at the level of the equations of motion.

5. Coherent state action for β-deformed $\mathcal{N} = 4$ SYM

To try to establish the AdS/CFT correspondence in the present deformed case let us now turn to gauge theory and address the question of which gauge-theory operators correspond to the fast-spinning strings discussed in the previous section and which are their anomalous dimensions.

The first important issue is whether in the deformed theory there exist subsets of relevant single-trace scalar operators which are closed under the renormalization group flow. A brief look at the Feynman rules in the β-deformed theory combined with the fact that the deformation preserves the three Cartan generators of the SO(6) R-symmetry group of $\mathcal{N}=4$ SYM leads one to the conclusion that at the 1-loop level all sectors closed in the original undeformed theory remain closed in the presence of the deformation. Given that the deformation does not introduce new types of interactions, this observation may be extended to higher loops as well. In the following we will be concerned in detail only with the operators built out of the two holomorphic scalar fields, i.e. $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2})+\cdots$. We will call this $su(2)_\beta$ *or "2-spin"* sector, by analogy with the $su(2)$ sector of the $\mathcal{N}=4$ SYM to which it reduces when the β-deformation is switched off. The sector of operators built out of the three holomorphic scalar fields will be called the $su(3)_\beta$ sector.

The dilatation operator of any field theory with matrix degrees of freedom can be represented as a hamiltonian of a spin chain acting on states in the spin chain Hilbert space. Indeed, for single-trace composite operators the matrix structure of the field theory defines a 1-dimensional lattice structure can be put into one-to-one correspondence with the lattice structure of the spin chain states. Then any operator acting on field theory composite operators has a (unique, up to symmetries) representation as an operator acting on the spin chain states. In the planar limit all such spin-chain operators are local, i.e. act on adjacent spin chain sites. An example is provided by the dilatation operator (see, e.g., [25, 26]). Constructed in perturbation theory, the L-loop (order λ^L) term in this operator acts on at most $L+1$ spin chain sites. At 1-loop this becomes a nearest-neighbour interaction, and given such interaction term it turns out to be very easy to engineer a field theory which has it as its dilatation operator in some sector.

An interesting property of the spin chain hamiltonians in various sectors of $\mathcal{N}=4$ SYM theory is that they are integrable. Remarkably, it turns out [37, 38] that the spin chain appearing in the $su(2)_\beta$ and $su(3)_\beta$ sectors of the β-deformed $\mathcal{N}=4$ SYM theory are also integrable for an arbitrary complex (in $su(2)_\beta$ case) and real (in $su(3)_\beta$ case) deformation parameter, respectively.

Before embarking on a detailed analysis of the properties of the spin chain for the $su(2)_\beta$ sector in this section we will first convince ourselves, following [15, 16], that its hamiltonian [37] can indeed be related to the world-sheet string theory in the β-deformed background. Explicitly, we shall construct the effective 2d action for the low energy "macroscopic" excitations of the spin chain in the special scaling limit when the length of the spin chain is taken to be large with $\tilde{\lambda}=\lambda/J^2$, J_1/J_2 *and* βJ (cf. (3.5)) kept fixed,

$$L\equiv J\to\infty\,,\qquad \tilde{\lambda}=\frac{\lambda}{J^2}=\text{fixed}\,,\qquad \beta J=\frac{\tilde{\beta}}{\sqrt{\tilde{\lambda}}}=\text{fixed}\,, \tag{5.1}$$

and show that it matches the string sigma model action (4.21) expanded in the same limit. As in the undeformed case, this implies the remarkable matching between the leading terms in the corresponding string energies and the 1-loop gauge-theory anomalous dimensions.

In appendix A we shall review the structure of the spin chain [37] for a 3-parameter family of deformations of $\mathcal{N} = 4$ SYM which contains as a special case the present β-deformed superconformal theory. There are several ways of writing the corresponding spin chain hamiltonian which are unitary equivalent. Here we find it convenient to present the hamiltonian (A.10) in the following form

$$\begin{aligned} \mathrm{H} = \frac{|h|^2\lambda}{(4\pi)^2}\sum_{l=1}^{J}\Big[&-\big(\boldsymbol{\sigma}_l^x\otimes\boldsymbol{\sigma}_{l+1}^x+\boldsymbol{\sigma}_l^y\otimes\boldsymbol{\sigma}_{l+1}^y+\boldsymbol{\sigma}_l^z\otimes\boldsymbol{\sigma}_{l+1}^z-\mathbb{1}_l\otimes\mathbb{1}_{l+1}\big)+\\ &+(1-\cosh 2\pi\kappa_d)\big(\boldsymbol{\sigma}_l^z\otimes\boldsymbol{\sigma}_{l+1}^z-\mathbb{1}_l\otimes\mathbb{1}_{l+1}\big)+\\ &+(1-\cos 2\pi\beta_d)\left(\boldsymbol{\sigma}_l^x\otimes\boldsymbol{\sigma}_{l+1}^x+\boldsymbol{\sigma}_l^y\otimes\boldsymbol{\sigma}_{l+1}^y\right)+\\ &+\sin 2\pi\beta_d\left(\boldsymbol{\sigma}_l^x\otimes\boldsymbol{\sigma}_{l+1}^y-\boldsymbol{\sigma}_l^y\otimes\boldsymbol{\sigma}_{l+1}^x\right)\Big]\,. \end{aligned} \tag{5.2}$$

Here $\boldsymbol{\sigma}^i$ are Pauli matrices, and β_d and κ_d are the deformation parameters from (2.2). This is a hamiltonian of a ferromagnetic XXZ spin chain [57] with broken parity invariance.[19] In the conformal limit the coefficient $|h|^2$ is related to the deformation parameter by the condition (2.5), i.e. h in general is not equal to 1 if β is complex. However, for the purpose of comparison with string theory we may set here $|h|^2 = 1$. Indeed, as discussed in section 3, we are interested in the limit when J (angular momentum or the spin chain length) is large while βJ (and λ/J^2) stays finite, i.e. the deformation parameter β is scaled to zero or $|e^{i\pi\beta}| \to 1$. We will thus ignore the factor $|h|^2$ in H in the following.[20]

An alternative spin chain (with parity-invariant hamiltonian) was discussed for the same theory in [38]; it is obtained from the one above by setting $\beta_d = 0$ in H while at the same time twisting the boundary conditions by β_d. The first line in (5.2) is the Heisenberg XXX spin chain hamiltonian of the $su(2)$ sector of the $\mathcal{N} = 4$ SYM [20] while the other terms arise due to the β-deformation. It is possible to remove the last term by performing a position-dependent unitary transformation [38].

The construction of the corresponding effective action in the continuum limit proceeds in the standard way (see, e.g., [59]). The first step is to rewrite the spin chain path integral as an integral over the overcomplete basis of coherent states

$$Z = \int [dn]\; e^{iS[n]}\,, \qquad S = \int dt\; \left(L_{WZ} - \langle n|\mathrm{H}|n\rangle\right), \tag{5.3}$$

where the first term L_{WZ} is the analog of the $p\dot{q}$ term in a phase space action coming from the Berry phase $\langle n|i\partial_t|n\rangle$.

In general, one chooses coherent states based on symmetry of the discrete hamiltonian one is interested in.[21] In our present case the (hidden) symmetry of the chain (5.2) is

[19] The parity-invariant XXZ spin chain with $\cosh 2\pi\kappa_d = 3$ appears as a 1-loop dilatation operator of $\mathcal{N} = 2$ SYM theory [58].

[20] Ignoring the factor of $|h|^2$ in H will also be consistent in section 8 where we will study $1/J$ corrections. Indeed, the equation (2.5) implies that $|h|^2 = 1 + \mathcal{O}(1/J^2)$ and thus will be subleading also at that order.

[21] The coherent states usually parametrize the coset G/H where G is a symmetry of the hamiltonian and H is a symmetry of the vacuum.

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$\widehat{\mathrm{SU}(2)}_{q_d}$ — the central extension of the quantum deformation of $su(2)$ (where q_d is defined in (2.7)), so one might be tempted to use the coherent state of the fundamental representation of this group.

It turns out to be sufficient for our purposes to use the same standard SU(2) coherent states parametrized by a unit vector $\mathbf{n}$ at each site l as in the undeformed case,

$$\langle n|\boldsymbol{\sigma}_i|n\rangle = n_i\,, \tag{5.4}$$

$$\sum_{i=1}^{3} n_i^2 = 1\,, \qquad \mathbf{n} = (\sin 2\theta\cos 2\eta, \sin 2\theta \sin 2\eta, \cos 2\theta)\,. \tag{5.5}$$

Then the Wess-Zumino term in the resulting coherent-state action will be the same as in the undeformed case, i.e.

$$L_{WZ} = \sum_{l=1}^{J} \mathcal{L}_{WZ}(\mathbf{n}_l, \dot{\mathbf{n}}_l)\,, \qquad \mathcal{L}_{WZ} = -\frac{1}{2}\int_0^1 ds\, \epsilon_{ijk} n^i \partial_s n^j \partial_t n^k = \cos 2\theta\, \dot\eta\,, \tag{5.6}$$

where, as usual, s is an auxiliary variable needed to put L_{WZ} in a manifestly SU(2)-invariant form.

Computing the expectation value of the hamiltonian (5.2) we would like to be able to take a continuum limit and define a semiclassical approximation for the resulting path integral, dominated by the classical action. For this to be possible, we need, as in the undeformed case [15, 16], to be able to put the action in (5.3) in the form with the factor J appearing in front of it, so that $1/J$ plays the role of an effective Planck's constant. Introducing the 1-d lattice spacing $a = 2\pi/J$ and defining $\mathbf{n}(\sigma)$ with $\mathbf{n}_l = \mathbf{n}(al)$ ($l = 1, 2, \ldots, J$) we conclude that we indeed need to fix $\tilde\lambda$ and βJ or $\tilde\beta$ in (5.1) (i.e. to assume $\beta \sim a$) to be able to define this scaling limit.

We then get for the continuous limit of the expectation values of different terms in (5.2):

- the first line in (5.2) gives the same as the continuum limit of the Heisenberg XXX hamiltonian:

$$\frac{1}{2}a^2\,(\mathbf{n}')^2 + \frac{1}{24}a^4\,(\mathbf{n}'')^2 + \cdots = 2a^2\left(\theta'^2 + \sin^2 2\theta\, \eta'^{\,2}\right) + O(a^4)\,, \tag{5.7}$$

- the second line in (5.2):

$$(n_z^2 - 1) - \frac{1}{2}a^2 (n_z')^2 + \cdots = -\sin^2 2\theta + O(a^2) \tag{5.8}$$

- the first term on the third line in (5.2):

$$(n_x)^2 + (n_y)^2 - \frac{1}{2}a^2\left[(n_x')^2 + (n_y')^2\right] + \cdots = \sin^2 2\theta + O(a^2)\,, \tag{5.9}$$

- the second term on the third line in (5.2):

$$a\left(n_x n_y' - n_y n_x'\right) - \frac{1}{2}a^3 n_x' n_y'' + \cdots = 2a \sin^2 2\theta\, \eta' + O(a^3)\,. \tag{5.10}$$

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Combining the leading order terms in (5.7), (5.8), (5.9) and (5.10) in the continuum limit $a = 2\pi/J \to 0$ we get

$$\langle n|\mathrm{H}|n\rangle = J\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\mathcal{H}_1 + \cdots\,, \tag{5.11}$$

$$\begin{aligned}\mathcal{H}_1 &= \frac{\lambda}{2J^2}\left(\theta'^2 + \sin^2 2\theta\,\eta'^{\,2}\right) + \frac{\lambda}{16\pi^2}(\cosh 2\pi\kappa_d - \cos 2\pi\beta_d)\sin^2 2\theta + \\ &\quad + \frac{\lambda}{4\pi J}\sin 2\pi\beta_d\;\sin^2 2\theta\,\eta'\,. \end{aligned}\tag{5.12}$$

It is easy to see that, as in the undeformed case, $\mathbf{n}_0 = (0,0,1)$ corresponds again to the vacuum of the hamiltonian (5.2) represented by the BPS operator $\mathrm{Tr}\,\Phi^J$. Indeed, H vanishes for $\theta = 0$, $\eta = 0$.

As follows from (5.12), to be able to define an effective action for which $\langle n|\mathrm{H}|n\rangle$ will be small at large J we thus need to assume that κ_d and β_d should be taken to zero with $J\kappa_d$ and $J\beta_d$ kept fixed.[22] Then,

$$\begin{aligned}\mathcal{H}_1 &= \frac{\lambda}{2J^2}\left[\theta'^2 + \sin^2 2\theta\left(\eta' + \frac{1}{2}\beta_d J\right)^2 + \frac{1}{4}(\kappa_d J)^2\sin^2 2\theta\right] \\ &= \frac{1}{2}\tilde{\lambda}\left(\theta'^{\,2} + \sin^2 2\theta\left|\eta' + \frac{1}{2}\beta J\right|^2\right). \end{aligned}\tag{5.13}$$

This is exactly the same as the anisotropic Landau-Lifshitz hamiltonian (4.27) we found above on the string theory side.[23] Also, the time-derivative dependent WZ term in the resulting action is the same as in the undeformed case, i.e. the same as in (4.21), so the two effective actions are exactly the same.

The potential $\sin^2 2\theta$ term has its absolute minimum at BPS state $\mathbf{n}_0 = (0,0,1)$ or $\theta = 0, \eta = 0$, and the spectrum of small fluctuations near this BPS vacuum found from the Landau-Lifshitz action (4.21) is the analog of the BMN spectrum with energies scaling as

$$E_n = \frac{\lambda}{2J^2}\left|n + \frac{1}{2}\beta J\right|^2. \tag{5.14}$$

We shall obtain the same spectrum directly from the Bethe ansatz for the spin chain in the next section.

6. Bethe Ansatz for the spectrum of anomalous dimensions of β-deformed $\mathcal{N} = 4$ SYM in the 2-spin sector

The hamiltonian (5.2) representing the one-loop dilatation operator in the $su(2)_\beta$ sector is integrable via the Bethe ansatz for all values of its parameters; the Bethe equations and

[22] If β_d and κ_d are kept finite at large J as would be for *discrete* values of the deformation parameters, (as, e.g., in the case of $\mathcal{N} = 2$ SYM discussed in [58]) the energies of spin-chain states will get large shifts proportional to J coming from $\sin^2\theta$ potential term and matching onto string states will not be possible.

[23] Note that, as was expected from the string-side discussion, we did not obtain any correction to the coefficient of θ'^2.

the cyclicity condition[24] are (in this section we use the notation $L \equiv J,\ M \equiv J - J_1 = J_2$)

$$e^{-2\pi i \beta_d L} \left[\frac{\sin(2\pi\kappa_d(\nu_k + i/2))}{\sin(2\pi\kappa_d(\nu_k - i/2))}\right]^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{\sin(2\pi\kappa_d(\nu_k - \nu_j + i))}{\sin(2\pi\kappa_d(\nu_k - \nu_j - i))}\,;$$

$$e^{-2\pi i \beta_d M} \prod_{k=1}^{M} \frac{\sin(2\pi\kappa_d(\nu_k + i/2))}{\sin(2\pi\kappa_d(\nu_k - i/2))} = 1\,, \tag{6.1}$$

where ν_k are rapidities of elementary excitations. The energy of a general state and the anomalous conformal dimension of the corresponding field theory operator $\mathrm{Tr}(\Phi_1^{L-M}\Phi_2^M) + \cdots$ is given by

$$E(\{\nu_k\}) = \sum_{k=1}^{M} \epsilon_k \qquad \text{with} \qquad \epsilon_k = \frac{\lambda}{8\pi^2}\frac{\sinh^2(2\pi\kappa_d)}{\sin(2\pi\kappa_d(\nu_k + i/2))\sin(2\pi\kappa_d(\nu_k - i/2))}\,. \tag{6.2}$$

The only difference of this Bethe ansatz from the usual one for the XXZ spin chain is in the presence of the additional β_d-dependent phase factors in the l.h.s. of the equations. The β_d dependence factors out of the Bethe equations and the cyclicity condition for generic κ_d. This is consistent with the treatment of the phase (β_d) of the deformation (2.7) in [38].

6.1 One-loop Long Chain

The Bethe equations can be used to compute anomalous dimensions of primary operators in the $su(2)_\beta$ sector for finite L just as it was done for the $su(2)$ sector of $\mathcal{N} = 4$ SYM [20]. For comparison with semiclassical string theory we are interested in studying long spin chains, $L \gg 1$. There are two different large L limits: (i) the "BMN limit": $L \to \infty$ with the number of excitations M kept fixed, and (ii) the "thermodynamic limit": $L \to \infty$ with the ratio M/L kept fixed. The BMN limit is used to compute conformal dimensions of near-BPS (BMN) operators [4], and the thermodynamic limit is used to find dimensions of operators dual to multi-spin string states [8].

Below we will use the above Bethe equations to derive the anomalous dimensions of BMN-type operators and the integral Bethe equations arising in the thermodynamic limit in the β-deformed theory. This will give us some understanding of the relation between (6.1) and the Bethe equations in the $su(2)$ sector of undeformed $\mathcal{N} = 4$ SYM. From the gauge theory standpoint, the anomalous dimensions of the BMN-type operators were computed in [60] using the technique of [61]. We will also derive integral Bethe equations appearing in the thermodynamic limit which later will be compared with the "string Bethe equations" which we will find from the string sigma model in the next section.

Since the details of the complex deformation parameter are somewhat more involved, we will begin by setting its imaginary part to zero and restore it later on.

Case of real β: according to (2.2) (see also appendix A) this deformation corresponds to

$$\kappa_d = 0\,. \tag{6.3}$$

[24] In the expression below we correct a misprint in [37].

In this limit the Bethe equations, the cyclicity condition and the energy of an individual excitation simplify considerably:

$$e^{-2\pi i\beta_d L}\left(\frac{\nu_k+i/2}{\nu_k-i/2}\right)^L = \prod_{i\neq k=1}^{M}\frac{\nu_k-\nu_j+i}{\nu_k-\nu_j-i}\,; \qquad e^{-2\pi i\beta_d M}\prod_{k=1}^{M}\frac{\nu_k+i/2}{\nu_k-i/2}=1\,, \tag{6.4}$$

$$\epsilon_k = \frac{\lambda}{8\pi^2}\frac{1}{\nu_k^2+1/4}\,. \tag{6.5}$$

We see that, up to the phase factors, these Bethe ansatz equations coincide with the usual ones for the XXX spin chain that describes the $su(2)$ subsector of the $\mathcal{N}=4$ SYM. Moreover, the expression of the energy in terms of the rapidities ν_k remains the same. It is clear, therefore, from (6.4) that to describe long spin chains we should rescale the rapidities

$$\nu_k = Lx_k\,, \tag{6.6}$$

and expand the equations in powers of $1/L$. Then these Bethe equations and the cyclicity condition reduce to the equations associated to the XXX Heisenberg chain [9] with the mode and momentum number shifted in a β_d -dependent way:

$$\frac{1}{x_k} = \frac{2}{L}\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{1}{x_k-x_j}+2\pi(n_k+\beta_d L)\,; \qquad \sum_{k=1}^{M}\frac{1}{x_k}=2\pi(m+\beta_d M)L\,. \tag{6.7}$$

$$\epsilon_k = \frac{\lambda}{8\pi^2L^2}\frac{1}{x_k^2}\,. \tag{6.8}$$

It is easy to verify that the existence of both the BMN and the thermodynamic limits requires that $\beta_d L$ should remain finite as $L\to\infty$, in a nice agreement with the discussions in section 3 and 5.

The equation (6.7) implies that *all* the solutions of the Bethe equations in the large L limits relevant for the undeformed $\mathcal{N}=4$ SYM theory can be easily translated into the theory deformed by a real β.

A simple illustrative example is that of the BMN-type operators – operators for which the number of excitations is kept finite as the length of the chain is taken to infinity. In this case the first term on the right hand side in the first equation (6.7) becomes subleading and the scaled rapidities x_k are

$$x_k = \frac{1}{2\pi(n_k+\beta_d L)} \tag{6.9}$$

while, in spite of the deformation, the cyclicity constraint becomes the same one as in the $su(2)$ sector of $\mathcal{N}=4$ SYM

$$\sum_{k=1}^{M} n_k = Lm\,. \tag{6.10}$$

Combining all the ingredients we find the anomalous dimension of BMN-type operators

$$E(n_1,\ldots,n_M) = \frac{\lambda}{2L^2}\sum_{k=1}^{M}(n_k+\beta_d L)^2 = \frac{1}{2}\lambda\sum_{k=1}^{M}\left(\frac{n_k}{L}+\beta_d\right)^2\,. \tag{6.11}$$

This agrees with the leading order term in the expansion of the energies of the string states in the plane wave limit of the deformed background found in [34] (see also [60]) and confirms the identification of the gauge theory parameter β_d with the string theory quantity $\beta\,|_{\sigma_d=0}$.

To derive the integral Bethe equations arising in the thermodynamic limit of the β-deformed theory we proceed in the same way as in the $\beta = 0$ case treated in [9]. Introducing the distribution density

$$\frac{1}{L}\sum_{k=1}^{M} f(x_k) = \int_C \mathrm{d}\xi\, \rho(\xi) f(\xi)\,; \qquad \rho(\xi) = \frac{1}{L}\sum_{k=1}^{M} \delta(\xi - x_k)\,, \tag{6.12}$$

with the support on a discrete union of n contours $C = C_1 \cup C_2 \ldots \cup C_n$, we obtain the integral Bethe equations

$$2\,⨍_C \mathrm{d}\xi\, \frac{\rho(\xi)}{x-\xi} = \frac{1}{x} - 2\pi\,(n_k + \beta_d L)\,, \qquad x \in C_k\,, \tag{6.13}$$

$$\int_C \mathrm{d}\xi\, \frac{\rho(\xi)}{\xi} = 2\pi\,(m + \beta_d M)\,; \qquad \int_C \mathrm{d}\xi\, \rho(\xi) = \frac{M}{L}\,. \tag{6.14}$$

The energy of a general state and the anomalous dimension of the corresponding field theory operator are expressed in terms of the density as follows

$$E = \frac{\lambda}{8\pi^2 L^2}\int_C \mathrm{d}\xi\, \frac{\rho(\xi)}{\xi^2}\,. \tag{6.15}$$

The equations (6.13)-(6.15) differ from the integral Bethe equations of the $su(2)$ sector of $\mathcal{N} = 4$ SYM [9] *only by the shifts of the mode numbers and the momentum number.* Therefore, *all* the solutions found in the $\mathcal{N} = 4$ SYM case can be easily translated into the theory deformed by a real β. In [17] all solutions of the above integral Bethe equation in the $\beta = 0$ theory have been constructed in terms of (noncompact) Riemann surfaces endowed with meromorphic differentials with integer periods. These periods over compact cycles are given by differences of mode numbers and as such are *invariant* under the shift in (6.13). The only period sensitive to this shift is the one over the noncompact cycle of the Riemann surface.

Case of complex β**:** while qualitatively similar, this case is a little bit more involved because we now have to also keep track of κ_d. It is not difficult to see from (6.1) that, for large L, κ_d gives a nontrivial contribution to the Bethe equations if $\kappa_d L$ is kept fixed in the large L limit. This implies in particular that $\tilde{q}_d = q_d^L = \exp(-2\pi\kappa_d L)$ and $\tilde{\sigma}$ in (2.26), (3.5) is kept finite on the string theory side.

To analyze the Bethe equation in this limit we find it convenient to make the following change of the rapidities ν_k

$$\tan(2\pi\kappa_d\,\nu_k) = 2\tanh(\pi\kappa_d)\,L\,x_k\,. \tag{6.16}$$

If $\kappa_d \to 0$ then this redefinition becomes equivalent to the rescaling (6.6) used to analyze the real β case.

Written in terms of the new parameters x_k, the Bethe equations (6.1) and (6.2) take the following form

$$e^{-2\pi i\beta_d L}\left[\frac{Lx_k+i/2}{Lx_k-i/2}\right]^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{x_k-x_j+\frac{i}{L}\frac{\tanh(2\pi\kappa_d)}{2\tanh(\pi\kappa_d)}(1+4\tanh^2(\pi\kappa_d)L^2x_kx_j)}{x_k-x_j-\frac{i}{L}\frac{\tanh(2\pi\kappa_d)}{2\tanh(\pi\kappa_d)}(1+4\tanh^2(\pi\kappa_d)L^2x_kx_j)}\,;$$

$$e^{-2\pi i\beta_d M}\prod_{k=1}^{M}\frac{Lx_k+i/2}{Lx_k-i/2} = 1\,; \tag{6.17}$$

$$E(\{\nu_k\}) = \sum_{k=1}^{M}\epsilon_k \quad \text{with} \quad \epsilon_k = \frac{\lambda}{8\pi^2}\,\frac{\cosh^2(2\pi\kappa_d)(1+4\tanh^2(\pi\kappa_d)L^2x_k^2)}{L^2x_k^2+1/4}\,. \tag{6.18}$$

Now the expansion of these equations in powers of $1/L$ is straightforward. Taking the logarithm, we obtain the following system which also describes subleading $1/L$ corrections to the Bethe equations[25]

$$\frac{1}{x_k} = \frac{2}{L}\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{1+(2\pi\kappa_d L)^2x_kx_j}{x_k-x_j}+2\pi(n_k+\beta_d L)\,; \qquad \sum_{k=1}^{M}\frac{1}{x_k} = 2\pi(m+\beta_d M)L\,; \tag{6.19}$$

$$\epsilon_k = \frac{\lambda}{8\pi^2L^2}\left[\frac{1}{x_k^2}+(2\pi\kappa_d L)^2\right] = \frac{\lambda}{8\pi^2L^2}\,\frac{1}{x_k^2}+\frac{1}{2}\lambda\kappa_d^2\,. \tag{6.20}$$

Recall that both $\beta_d L$ and $\kappa_d L$ should be kept fixed in the large L limit.

We see that the energies of individual excitations differ from the real β case only by a constant shift $\frac{1}{2}\lambda\kappa_d^2$. Moreover, in the BMN limit ($L\to\infty$ with $M/L\to 0$) the equations (6.19) coincide with the equations (6.7) describing the BMN limit in the real β case. Therefore, the anomalous dimensions of the BMN operators are given again by (6.11) with the constant shift of energies contributing an extra term $\frac{1}{2}\lambda\kappa_d^2 M$

$$E = \sum_{k=1}^{M}\epsilon_k = \frac{\lambda}{2L^2}\sum_{k=1}^{M}\left[(n_k+\beta_d L)^2+(\kappa_d L)^2\right] = \frac{1}{2}\lambda\sum_{k=1}^{M}\left|\frac{n_k}{L}+\beta\right|^2. \tag{6.21}$$

This expression indeed reproduces the leading order term in the expansion of the energies of string states in the plane wave limit of the deformed background which were derived in [34] (see also [60]).[26]

Similarly to the case of real β parameter, the expression (6.21) can be obtained from the one in the undeformed case by shifting the mode numbers by the (now complex) deformation parameter β. It is, however, clear that the real and the imaginary parts of $\beta = \beta_d - i\kappa_d$ enter differently into the Bethe equations and so the eigenvalues of the hamiltonian depend on both the real and imaginary parts of $n_k+\beta L$ but not only on $|n_k+\beta L|$; this is in a qualitative agreement with our conclusion based on the string world sheet analysis.

[25] There exists a potential for anomalies due to roots with spacing of order $1/L$ [27, 62]. We will return to this in section 8.

[26] The operators dual to these string states can be easily reconstructed using the details of the Bethe ansatz. It turns out that the 2-impurity operators are identical to those in the undeformed theory.

It is not difficult to obtain the integral Bethe equation describing the thermodynamic limit of the spin chain for complex β. The same equation should be possible to derive from the Landau-Lifshitz action for fast strings we found in section 4.

6.2 Comments on higher loop orders

Let us now make few comments on higher-loop corrections to the dilatation operator of the β-deformed theory and possible extension of the associated Bethe ansatz. Let us first recall that in the $\beta = 0$ case the 2-loop dilatation operator contains only 2-spin (nearest neighbour and next to nearest neighbour) interactions [21]

$$\mathrm{H}_2 = \frac{\lambda^2}{(4\pi)^4}\sum_{l=1}^{L}(-3 - \boldsymbol{\sigma}_l\cdot\boldsymbol{\sigma}_{l+2} + 4\boldsymbol{\sigma}_l\cdot\boldsymbol{\sigma}_{l+1})\,, \tag{6.22}$$

and is expected to be part of an integrable spin chain [14, 24]. It can be indeed embedded [14] into the integrable Inozemtsev [63] spin chain (with only 2-spin interactions in the hamiltonian) but the correct $\mathcal{N} = 4$ SYM dilatation operator in the $su(2)$ sector is different, containing also 4-spin and higher interactions at three and higher loops [21, 25]. The conjecture form of the corresponding asymptotic (L large compared to loop order) Bethe ansatz was suggested in [24]

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i}\,, \tag{6.23}$$

were p_k are the magnon momenta and the analogs of the rapidities $\nu_k = \frac{1}{2}\cot\frac{p_k}{2}$ are [24][27]

$$\varphi_k \equiv \varphi(p_k)\,, \qquad \varphi(p) = \frac{1}{2}\cot\frac{p}{2}\,\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\,, \tag{6.24}$$

with $E = \sum_{k=1}^{M}\left(\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}} - 1\right)$.

Turning now to the β-deformed theory, a natural series of questions are: (i) Which is the structure of the 2-loop dilatation operator? (ii) Is it integrable, e.g., is it possible to embed into some integrable Inozemtsev-type spin chain with a Bethe ansatz, whose solution, rewritten as a series expansion in the 't Hooft coupling, reproduces both the 1-loop and 2-loop anomalous dimensions?[28] (iii) Is it possible to generalize the Bethe ansatz (6.23), (6.24) to the case of non-zero deformation β?

[27] In the case of the Inozemtsev chain $\varphi(p) = \frac{1}{2}\cot\frac{p}{2} + 2\sum_{n=1}^{\infty}\frac{t^n \sin p}{(1-t^n)^2+4t^n\sin^2\frac{p}{2}}$. To match the 2-loop dilatation operator [14] t should be related to the 't Hooft coupling as $\frac{\lambda}{16\pi^2} = \sum_{n=1}^{\infty}\frac{t^n}{(1-t^n)^2}$.

[28] Let us note that from a practical standpoint of the calculation of anomalous dimensions, the integrability of the 2-loop dilatation operator is not crucial, as one can always treat the 2-loop corrections as a perturbation of the 1-loop integrable hamiltonian; then $\delta E = \langle E_{(1)}|\mathrm{H}_{2-\mathrm{loop}}|E_{(1)}\rangle$, where $|E_{(1)}\rangle$ stands for an eigenstate of the 1-loop hamiltonian with energy $E_{(1)}$.

Starting with the Feynman diagrams for the theory (2.1), (2.6) one observes that the 2-loop dilatation operator receives two types of contributions. The first type corrects at order λ^2 the nearest neighbour interactions already appearing at the 1-loop level in (5.2): they are given by Feynman diagrams involving only two fields from the operator whose renormalization we are studying. The second type involves qualitatively new structures which can be interpreted as 3-spin interaction terms. They are rather involved and we will not present them here in full generality. Fortunately, the important features can be extracted already from only partial 2-loop calculations.

Let us first discuss the pure phase deformation, $\beta \in \mathbb{R}$. In this case, an observation of [38] makes it easy to construct the spin chain hamiltonian representing the 2-loop dilatation operator. First, we note that by a position dependent phase transformation the 1-loop hamiltonian (5.2) can be mapped into the hamiltonian of the XXX Heisenberg chain, i.e. into the 1-loop dilatation operator in the $su(2)$ sector of $\mathcal{N}=4$ SYM. This can be formally implemented as a unitary transformation generated by

$$\mathcal{U}(\varphi) = \exp\left[-\frac{1}{2}i\varphi\sum_{k=1}^{L} k\,(1-\boldsymbol{\sigma}_k^z)\right], \tag{6.25}$$

which acts on the generators of SU(2) as

$$\mathcal{U}(\varphi)\boldsymbol{\sigma}_l^{\pm}\mathcal{U}^\dagger(\varphi) = e^{\pm il\varphi}\boldsymbol{\sigma}_l^{\pm}\,, \qquad \mathcal{U}(\varphi)\boldsymbol{\sigma}_l^{z}\mathcal{U}^\dagger(\varphi) = \boldsymbol{\sigma}_l^{z}\,. \tag{6.26}$$

Second, in [37] it was noticed that the 4-scalar interaction in (2.6) coincides (up to addition of the identity operator) with the 1-loop hamiltonian. As a result, one is able to argue that the 2-loop spin chain hamiltonian for the deformed theory with $\beta \in \mathbb{R}$ is the same as the 2-loop spin chain hamiltonian of the $\mathcal{N}=4$ SYM (6.22), unitary-transformed with $\mathcal{U}^{-1}$.

Interpreting the Bethe equations as the cyclicity conditions for the magnon excitations, it then follows that the effect of this $\mathcal{U}^{-1}$ transformation should be to *twist the boundary conditions* of the spin chain. This suggests the integrable extension of the 2-loop dilatation operator in the $\beta \in \mathbb{R}$-deformed $\mathcal{N}=4$ SYM theory should be the same spin chain as in the absence of the deformation but with twisted boundary conditions. The corresponding asymptotic Bethe ansatz equations should then be given by the following modification of (6.23)

$$e^{i(p_k-2\pi\beta_d)L} = e^{-2\pi i\beta_d L}e^{ip_kL} = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{\varphi_k-\varphi_j+i}{\varphi_k-\varphi_j-i}\,. \tag{6.27}$$

In this case the 1- and 2-loop matching between gauge and string theory observed in undeformed case [14, 16, 17] should extend also to the real β deformation case. Indeed, the subleading order λ^2 correction (4.28) to the string reduced (Landau-Lifshits) sigma model was found to depend on β_d only through the combination $\eta' + \frac{1}{2}\beta_d J$, and thus can be absorbed by redefining $\eta \to \tilde{\eta} = \eta + \frac{1}{2}\beta_d J\sigma$, where the new field satisfies $\tilde{\eta}(\tau,\sigma+2\pi) = \tilde{\eta}(\tau,\sigma) + \pi\beta_d J$, i.e. has twisted boundary condition.

It is possible that this $\mathcal{U}^{-1}$ transformation extends to arbitrary number of loops and then conjecture that that the anomalous dimensions of all operators in the 2-spin sector are given by the the Bethe-like equations as in [28] modified by the inclusion of the phase $\exp(-2\pi i\beta_d)$ in the term describing the expected phase shift $\exp(ip_k L)$ of a magnon transported along the chain (see also the next section).

The case of general complex deformation parameter is by far less clear. One very optimistic guess would be that there exists an integrable spin chain like the Inozemtsev chain in which the exchange operator $P_{ij} = I - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ is replaced by the 2-site hamiltonian in (5.2) or (A.9).[29] It seems unlikely, however, that the explicit calculations will support this. Indeed, looking at the Feynman diagrams, a potential problem is the appearence of 3-spin interaction terms which cannot be put into the form of a long range 2-spin interaction. Using the field-theory ingredients spelled out in [73] and (2.6), (2.7) we can isolate such a term:

$$(q_d^2 - 1)(\boldsymbol{\sigma}^+ \otimes \boldsymbol{\sigma}^- + \boldsymbol{\sigma}^- \otimes \boldsymbol{\sigma}^+) \otimes (\mathbb{1} - \boldsymbol{\sigma}^z)\,. \tag{6.28}$$

The difference between the couplings of the fields Φ_i for zero and non-zero κ_d prevents the term with opposite signs of $\boldsymbol{\sigma}^z$ from being generated and thus this term seems to survive as a genuine 3-spin interaction. The status of the integrability of such spin chains with next-to-nearest neighbour interactions is not clear. Certain 3-spin deformations of the XXZ chain were analyzed in [74], but none of the hamiltonians considered there contain (6.28). Even though no exhaustive search has been carried out, we are tempted to expect that the 2-loop spin chain hamiltonian of the deformed theory with complex β does not have an integrable extension.

There exists at least one natural reason to expect that the integrability of the dilatation operator of the β-deformed $\mathcal{N} = 4$ SYM is spoiled at some number of loops if $Im(\beta) \neq 0$. The construction of the supergravity dual of this type of deformation required [34] the use of S-duality transformations. Even though the resulting string coupling is small (due to the use of the *two* S-duality transformations) this seems to imply that non-interacting strings in the deformed background should have a knowledge of interactions in the undeformed theory. We, however, do not expect the dilatation operator of the $\mathcal{N} = 4$ SYM theory be integrable in such finite N regime. The fact that the dilatation operator in the 3-spin $(\mathrm{Tr}\,\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3} + \cdots)$ sector of the deformed theory is not integrable already at the 1-loop level if $\sigma_d \neq 0$ may be also an indication of this.

7. String Bethe equations

In this section we shall use the Lax representation for strings on the Lunin-Maldacena background (2.8) with real $\tilde{\beta} \equiv \tilde{\gamma}$ recently found in [41] to derive the string Bethe equations in the corresponding "$su(2)_\gamma$" subsector of the model (in this section $\kappa_d = 0$, i.e. $\beta_d = \gamma$, see (2.2), (2.26). In this subsector we consider strings moving on the deformed background $R \times S^3_\gamma$, described by the action (4.2). By making the T-duality transformation on ϕ_1, a

[29] In [64] it was conjectured that the Haldane-Shastry chain (and then probably also the Inozemtsev chain) has an anisotropic extension.

shift of ϕ_2, followed by another T-duality on $\tilde{\phi}_1$, we get back to the standard action for a string on $R\times S^3$:

$$S=-\frac{\sqrt{\lambda}}{2}\int d\tau\int_0^{2\pi}\frac{d\sigma}{2\pi}\gamma^{\alpha\beta}\left[-\partial_\alpha t\partial_\beta t+\sum_{i=1}^{2}\left(\partial_\alpha\rho_i\partial_\beta\rho_i+\rho_i^2\partial_\alpha\tilde{\tilde{\phi}}_i\partial_\beta\tilde{\tilde{\phi}}_i\right)\right], \qquad (7.1)$$

where a double tilde over ϕ_i reflects the two T-duality transformations we performed.

It is not difficult to find the following on-shell relations between $\tilde{\tilde{\phi}}_i$ and ϕ_i [41]

$$\begin{aligned}\partial_\alpha\tilde{\tilde{\phi}}_1&=G\left(\partial_\alpha\phi_1-\tilde{\gamma}\,\rho_2^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_2\right),\\ \partial_\alpha\tilde{\tilde{\phi}}_2&=G\left(\partial_\alpha\phi_2+\tilde{\gamma}\,\rho_1^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_1\right).\end{aligned} \qquad (7.2)$$

By introducing the momenta p_i and $\tilde{\tilde{p}}_i$ conjugated to ϕ_i and $\tilde{\tilde{\phi}}_i$ respectively, the relations (7.2) can be rewritten in the following simple form

$$\tilde{\tilde{p}}_i=p_i\,,\qquad \tilde{\tilde{\phi}}{}_1'=\phi_1'+\gamma\,p_2\,,\qquad \tilde{\tilde{\phi}}{}_2'=\phi_2'-\gamma\,p_1\,, \qquad (7.3)$$

where $\gamma=\tilde{\gamma}/\sqrt{\lambda}$ is the deformation parameter that appears on field theory side. Taking into account that ϕ_i are angle variables, and integrating (7.3) over σ, we get the following twisted boundary conditions for the U(1) variables $\tilde{\tilde{\phi}}_i$ of the $R\times S^3$ model:

$$\begin{aligned}\phi_i(2\pi)-\phi_i(0)&=2\pi\,n_i\,,\quad n_i \text{ are integer winding numbers},\\ \tilde{\tilde{\phi}}_1(2\pi)-\tilde{\tilde{\phi}}_1(0)&=2\pi\,(n_1+\gamma\,J_2)\,,\qquad \tilde{\tilde{\phi}}_2(2\pi)-\tilde{\tilde{\phi}}_2(0)=2\pi\,(n_2-\gamma\,J_1)\,.\end{aligned} \qquad (7.4)$$

The relations (7.2), (7.3) and (7.4) imply that if ϕ_i solve equations of motion for strings on $R\times S^3_\gamma$ then $\tilde{\tilde{\phi}}_i$ solve those on $R\times S^3$ with the twisted boundary conditions (7.4) imposed on $\tilde{\tilde{\phi}}_i$, and vice versa. One can also check that if the Virasoro constraints are satisfied for strings on $R\times S^3_\gamma$ then they are satisfied for twisted strings on $R\times S^3$, and, therefore, the energy of a twisted string is equal to the energy of the corresponding string on $R\times S^3_\gamma$.

Since the Lax pair for strings on $R\times S^3_\gamma$ is closely related to the Lax pair for the sigma model on S^3, we recall the necessary facts about the latter model. We follow closely the discussion in [17] to simplify the comparison of the string Bethe equations for the string sigma model on $R\times S^3$ with the equations we will derive for the $R\times S^3_\gamma$ model.

We parameterize S^3 by unitary SU(2) matrices of the form:

$$g=\begin{pmatrix}X_1 & X_2\\ -X_2^* & X_1^*\end{pmatrix},\qquad \det g=|X_1|^2+|X_2|^2=1\,,\qquad X_i=\rho_i e^{i\tilde{\tilde{\phi}}_i}\,. \qquad (7.5)$$

The equations of motion for the string sigma model on $R\times S^3$ follow from the action (7.1) that can be cast in the form:

$$\mathrm{S}=\frac{\sqrt{\lambda}}{4\pi}\int \mathrm{d}\tau\mathrm{d}\sigma\gamma^{\alpha\beta}\left[\partial_\alpha t\partial_\beta t+\frac{1}{2}\mathrm{Tr}\Big(g^{-1}\,\partial_\alpha g\,g^{-1}\,\partial_\beta g\Big)\right].$$

Introducing the right current

$$R_\alpha=g^{-1}\,\partial_\alpha g\,,$$

the equations of motion for the matrix field g can be written in the form

$$\partial_\alpha(\gamma^{\alpha\beta}R_\beta) = 0\,. \tag{7.6}$$

They should be supplemented by the Virasoro constraints:

$$\partial_0 t\partial_0 t + \partial_1 t\partial_1 t + \frac{1}{2}\,\mathrm{Tr}\left(R_0^2 + R_1^2\right) = 0\,, \qquad \partial_0 t\partial_1 t + \frac{1}{2}\,\mathrm{Tr}(R_0R_1) = 0\,.$$

The equations (7.6) are equivalent to the zero curvature condition [65–67, 56]

$$[D_\alpha\,,\, D_\beta] = 0\,, \tag{7.7}$$

where the Lax operator depending on a spectral parameter x is defined as

$$D_\alpha = \partial_\alpha - \frac{R_\alpha^+}{2(x-1)} + \frac{R_\alpha^-}{2(x+1)} \equiv \partial_\alpha - \mathcal{A}_\alpha(x)\,. \tag{7.8}$$

Here the self-dual and anti-self dual projections of R_α are given by

$$R_\alpha^\pm = (P^\pm)_\alpha^{\ \beta} R_\beta\,, \qquad (P^\pm)_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} \mp \gamma_{\alpha\rho}\epsilon^{\rho\beta}\,. \tag{7.9}$$

To obtain the Lax representation for the $R\times S^3_\gamma$ model we perform the following gauge transformation of the Lax connection $\mathcal{A}_\alpha$[30]

$$\begin{aligned} &D_\alpha \to \mathcal{M}\,D_\alpha\,\mathcal{M}^{-1} = \partial_\alpha - \mathcal{R}_\alpha\,,\\ &\mathcal{R}_\alpha = \mathcal{M}\mathcal{A}_\alpha\,\mathcal{M}^{-1} - \mathcal{M}\,\partial_\alpha\mathcal{M}^{-1} \;=\; \hat{\mathcal{A}}_\alpha + \frac{i}{2}(\partial_\alpha\tilde{\tilde{\phi}}_2 - \partial_\alpha\tilde{\tilde{\phi}}_1)\,\sigma_3\,, \end{aligned} \tag{7.10}$$

where

$$\mathcal{M} = \begin{pmatrix} 0 & e^{\frac{i}{2}(\tilde{\tilde{\phi}}_2-\tilde{\tilde{\phi}}_1)} \\ e^{-\frac{i}{2}(\tilde{\tilde{\phi}}_2-\tilde{\tilde{\phi}}_1)} & 0 \end{pmatrix}, \qquad \mathcal{M}^{-1} = \mathcal{M}\,. \tag{7.11}$$

One can show that the new Lax connection $\mathcal{R}_\alpha$ depends on $\tilde{\tilde{\phi}}_i$ only through $\partial_\alpha\tilde{\tilde{\phi}}_i$ [41]. The local and periodic Lax connection for the $R\times S^3_\gamma$ model is now obtained by expressing the twisted angle variables $\tilde{\tilde{\phi}}_i$ of S^3 in terms of the angle variables ϕ_i of $R\times S^3_\gamma$ by using the relations (7.2).

To derive the string Bethe equations we need to analyze various asymptotic properties of the monodromy matrix $\mathrm{T}(x)$ which is defined as the path-ordered exponential of the spatial component $\mathcal{R}_1(x)$ of the Lax connection

$$\mathrm{T}(x) = \mathcal{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\,\mathcal{R}_1(x)\,, \tag{7.12}$$

and also of the quasi-momentum $p(x)$ defined by

$$2\cos p(x) = \mathrm{Tr}\ \mathrm{T}(x)\,. \tag{7.13}$$

[30]A similar idea was used in [68] to derive a local and periodic Lax connection for the hamiltonian of strings on $AdS_5\times S^5$.

It is well-known that if the Lax connection is periodic then the quasi-momentum $p(x)$ is conserved for any value of the spectral parameter x, and it plays an important role in the inverse scattering method [69].

The Lax connection has poles at $x = \pm 1$, and the analysis of the asymptotic behavior of the quasi-momentum around the poles does not, in fact, require any computation. All one needs to do is to notice that around the poles one can diagonalize the Lax connection $\mathcal{R}_1$ by means of a regular gauge transformation which depends only on $\partial_\alpha \tilde{\tilde{\phi}}_i$, and, therefore, is periodic. This gauge transformation diagonalizes the Lax connections both for the $R\times S^3_\gamma$ and the $R \times S^3$ models. Therefore, the asymptotic behavior of the quasi-momentum in both models is the same [17]

$$p(x) = -\frac{\pi\kappa}{x \pm 1} + \cdots \qquad x \to \mp 1\,, \tag{7.14}$$

where κ is the rescaled energy of the string solution

$$E = \sqrt{\lambda}\kappa\,, \qquad \kappa = -\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\,\gamma^{0\alpha}\partial_\alpha t\,. \tag{7.15}$$

In what follows, we set, without loss of generality, $\gamma^{\alpha\beta} = \mathrm{diag}(-1,1)$ and $t = \kappa\tau$.

The analysis of the asymptotic behavior of the quasi-momentum at infinity and zero is more involved because it does not vanish at large values of the spectral parameter x, and we present its details in the appendix B. The results of this analysis are summarized below:

$$p(x) = \pi\gamma L - \frac{2\pi(L-2M)}{\sqrt{\lambda}x} + \cdots\,, \quad x \to \infty\,, \tag{7.16}$$

$$p(x) = \pi\gamma(L-2M) + \frac{2\pi L}{\sqrt{\lambda}}\,x - 2\pi m + \cdots\,, \quad x \to 0\,, \tag{7.17}$$

where

$$L \equiv J = J_1 + J_2\,, \qquad M \equiv J_2\,,$$

and m is an integer quasi-momentum number.

Subtracting the singularities at $x \to \pm 1$, and the constant term at $x \to \infty$, we define the resolvent

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1} - \pi\gamma L\,. \tag{7.18}$$

Following [17], we assume that the resolvent is an analytical function on the complex plane of the spectral parameter x with a finite number of cuts C_i. Then, it admits the following spectral representation

$$G(x) = \int_C \mathrm{d}\xi\,\frac{\rho(\xi)}{x-\xi}\,, \qquad C = C_1 \cup C_2 \ldots \cup C_n\,, \tag{7.19}$$

where $\rho(\xi)$ is a positive spectral density.

The resolvent has the following asymptotic behavior at infinity and zero

$$
\begin{aligned}
G(x) &= \frac{2\pi}{\sqrt{\lambda}\,x}(E+2M-L)+\cdots\,, \qquad x\to\infty \\
G(x) &= -2\pi\gamma M-2\pi m-\frac{2\pi x}{\sqrt{\lambda}}(E-L)+\cdots\,, \qquad x\to 0\,.
\end{aligned} \tag{7.20}
$$

This asymptotic behavior leads to the following constraints imposed on the density

$$
\int_C \mathrm{d}\xi\,\rho(\xi) = \frac{2\pi}{\sqrt{\lambda}}(E+2M-L)\,, \tag{7.21}
$$

$$
-\frac{1}{2\pi i}\oint\frac{G(x)}{x}\mathrm{d}x = \int_C \mathrm{d}\xi\,\frac{\rho(\xi)}{\xi} = 2\pi\gamma M+2\pi m\,, \tag{7.22}
$$

$$
-\frac{1}{2\pi i}\oint\frac{G(x)}{x^2}\mathrm{d}x = \int_C \mathrm{d}\xi\,\frac{\rho(\xi)}{\xi^2} = \frac{2\pi}{\sqrt{\lambda}}(E-L)\,. \tag{7.23}
$$

Finally, the quasi-momentum obeys the unimodularity condition [17]

$$
p(x+i0)+p(x-i0) = -2\pi n_k\,, \qquad x\in C_k\,, \tag{7.24}
$$

that leads to the following singular integral equation for the spectral density

$$
G(x+i0)+G(x-i0) = 2\,⨍_C \mathrm{d}\xi\,\frac{\rho(\xi)}{x-\xi} = \frac{2\pi\kappa}{x-1}+\frac{2\pi\kappa}{x+1}-2\pi\gamma L-2\pi n_k\,, \qquad x\in C_k\,. \tag{7.25}
$$

Comparing equations (7.21), (7.22), (7.23) and (7.25) with those for the undeformed case [17], we see that they differ only by the γ-dependent shifts of the quasi-momentum number m and the mode numbers n_k. Therefore, the comparison of these equations with the thermodynamic Bethe equations for the γ-deformed spin chain we derived in section 6 repeats the comparison done in [17] for the undeformed case.[31] We conclude that the string Bethe equations coincide with the thermodynamic Bethe equations for the γ-deformed spin chain at the one-loop (λ) and two-loop (λ^2) orders. This constitutes a proof of two-loop agreement between the string and the gauge theory results in the $su(2)_\gamma$ subsector.

Moreover, it is straightforward to write down the quantum (i.e. discrete) version of the classical Bethe equations following [28], and assuming that the phase function appearing in the Bethe ansatz for the β-deformed spin chain coincides with the phase function of the asymptotic Bethe ansatz for the $su(2)$ subsector of $\mathcal{N}=4$ SYM [24].[32]

It is natural to expect that, as it was the case in the undeformed case [28], the Bethe equations for quantum strings will reproduce the $1/J$ correction to the BMN states. It would be interesting to check this by a direct string theory computation generalizing the one done in [71]. The string theory action can be obtained (up to the quartic order in string fields including fermions) from the string action on $AdS_5\times S^5$ by using the TsT transformation we discussed in sections 1 and 2. It would be also interesting to understand how the relation between the $su(2)$, $sl(2)$ and $su(1|1)$ subsectors of $\mathcal{N}=4$ SYM found in [29] is modified in the deformed case.

[31] To perform the comparison one should first rescale x and ξ: $(x,\,\xi)\to\frac{4\pi L}{\sqrt{\lambda}}\,(x,\,\xi)$.

[32] The phase function was derived in [24] assuming the validity of the BMN scaling. Let us note that a recent computation [70] of the 4-loop dilatation operator in a matrix model related to $\mathcal{N}=4$ SYM by dimensional reduction on S^3 suggests that the BMN scaling may be broken (in the matrix model) starting at 4-loop order.

8. Subleading $1/J$ corrections

Given the success in the comparison of the 1-loop spin chain and fast string expansion of the sigma model, it is natural to ask whether it can be extended to subleading terms in the large quantum number expansion. Then on the gauge theory / spin chain side we are interested in finite size corrections to the thermodynamic limit of the Bethe ansatz while on the string theory side we are interested in world sheet quantum corrections [8, 75, 27].

An interesting fact which emerged from the calculation of the leading $1/J$ correction term in the world sheet 1-loop correction to the energies of semiclassical circular string solutions in $AdS_5 \times S^5$ [75] is that they are reproduced by the 1-loop calculation in the Landau-Lifshitz theory, provided one uses the ζ-function regularization to compute the infinite sum over charactersitic frequencies [27].[33]

In the case of the β-deformed theory, the analysis of the 1-loop corrections to the string energies, while certainly doable, is somewhat complicated by the large number of fluxes in the supergravity background. In the following we will not perform the full string theory calculation; rather, we will follow the example of the undeformed theory and use the "fast string" expansion of the string sigma model. We will compute the 1-loop correction to the energy of circular 2-spin solution in this theory using the ζ-function regularization and compare it with the leading finite size correction to the thermodynamic limit of the Bethe Ansatz. We will find that, as in the case of the undeformed theory [27], the two expressions are in perfect agreement.

8.1 String sigma model side

In general, computing the quantum shift in the energy of some classical solution requires finding and summing up the frequencies of the fluctuations around it. The "fast string" limit of the sigma model has some simple classical solutions for which it is possible to do this analysis very explicitly.

The case of real deformation parameter is again very easy to analyze. As we have seen in section 4.2, the effect of such deformation is to twist the boundary conditions of the η field in the world sheet σ direction (see (4.21)). We, therefore, expect that in this case the frequencies of the fluctuation modes near the circular solution of section 4.1 or (4.19) should follow from the results of [27] for $\beta = 0$ theory by shifting the winding number m by $\frac{1}{2}\beta_d J$

$$E_1^{(1)} = \frac{\lambda}{2J^2}(m+\frac{1}{2}\beta_d J)^2 + \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}\left[n\sqrt{n^2-4(m+\frac{1}{2}\beta_d J)^2} - n^2 + 2(m+\frac{1}{2}\beta_d J)^2\right]. \tag{8.1}$$

This expectation is, of course, realized. We will recover the equations above as a limit of the frequencies in the presence of a complex deformation.

[33] The reason for that is that the relevant contribution comes essentially from the characteristic frequencies in the directions included in the Landau-Lifshitz action, while the role of other fluctuations in other bosonic and fermionic directions is to regularize the resulting infinite sum.

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The lagrangian (4.21) or (5.6), (5.13) expressed in terms of the unit vector $\vec{n}(\theta, \eta)$ in (5.5) is

$$L = L_{WZ} - \frac{\tilde{\lambda}}{8}\left[\mathbf{n}'^{\,2} - |\beta J|^2(n_z^2 - 1) + 2\beta_d J(n_x n_y' - n_y n_x')\right]. \tag{8.2}$$

Using that $\delta L_{WZ} = \frac{1}{2}\epsilon_{ijk}\delta n_i n_j \dot{n}_k$ leads to the equations of motion

$$\frac{1}{2}\epsilon_{ijk} n_j \dot{n}_k = -\frac{\tilde{\lambda}}{4}\left(n_i'' + v_i\right)_\perp \quad \longrightarrow \quad \dot{n}_i = \frac{\tilde{\lambda}}{2}\epsilon_{ijk} n_j \left(n_k'' + v_k\right) \tag{8.3}$$

where the vector v is given by

$$\mathbf{v} = J\left(-2\beta_d\, n_y', 2\beta_d\, n_x', J|\beta|^2 n_z\right). \tag{8.4}$$

and $\perp$ denotes the projection orthogonal to $\mathbf{n}$. When the deformation is removed, $\mathbf{v}$ vanishes and the equations (8.3) become the standard isotropic Landau-Lifshitz equations.

We have seen in sections 4 and 5 that the terms linear in σ derivatives are producing the twisting of the boundary conditions of the world sheet field η. Taking this standpoint (that is, removing the first two components of $\mathbf{v}$ and recalling that the winding number should then be shifted by $\frac{1}{2}J\beta_d$) we see that the equations above are a special case of the anisotropic Landau-Lifshitz equations, which are known to be integrable [56]. As was already noted above, this squares nicely with the lagrangian (8.2) being the continuum limit of the coherent state action for the integrable spin chain.

Let us now consider fluctuations near the solution (4.19) with $J_1 = J_2$. This is straightforward to do in terms of the two angular coordinates, but the result may be presented also in terms of $\mathbf{n}$ parametrization. If we consider the solution (cf. (4.19), (5.5))

$$\mathbf{n} = (\cos 2m\sigma,\, \sin 2m\sigma,\, 0), \tag{8.5}$$

then parameterizing the fluctuations as

$$\delta\mathbf{n} = (-\sin 2m\sigma\, A_1(\tau,\sigma),\, \cos 2m\sigma\, A_1(\tau,\sigma),\, A_2(\tau,\sigma)), \tag{8.6}$$

it is a simple exercise to linearize the equations of motion. All world sheet position dependence disappears and we are left with two coupled Schrödinger-type equations

$$\dot{A}_1 = \frac{\tilde{\lambda}}{2}\left[A_2'' + 4|m + \frac{1}{2}\beta J|^2 A_2\right], \qquad \dot{A}_2 = \frac{\tilde{\lambda}}{2}A_1''. \tag{8.7}$$

Introducing the mode expansion $A_s \propto \sum C_{s,n} e^{i\omega_n\tau + in\sigma}$ (the boundary conditions on the fluctuations are not twisted in this parametrization) we find the mode frequencies:

$$\omega_n = \frac{\tilde{\lambda}}{2} n\sqrt{n^2 - 4\left|m + \frac{1}{2}\beta J\right|^2}. \tag{8.8}$$

This result is very similar to the one in the absence of the deformation.[34] The energy shift is given by the sum of the frequencies (8.8). Applying then the ζ-function regularization

[34] It is interesting to note that while the initial field equations depended separately on the real and imaginary parts of the deformation β, the frequencies of the fluctuations happen to depend only on the modulus of complex β shifted by $2m/J$. Most likely this is not a general property but rather a special feature of the simple solution (4.19).

as in [27], we conclude that the 1-loop correction to the energy of the classical string solution (8.5) is given by the same expressions as in [27] with the winding number m replaced by by $|m + \frac{1}{2}\tilde{\beta}\mathcal{J}|$

$$E_1^{(1)} = E_{\rm reg} + E_{\rm fin}\,, \quad E_{\rm reg} = \frac{\lambda}{2J^2}\left[\sum_{n=1}^{\infty}\left(n^2 - 2\Big|m + \frac{1}{2}\beta J\Big|^2\right)\right]_{\rm reg} = \frac{\lambda}{2J^2}\Big|m + \frac{1}{2}\beta J\Big|^2 \quad (8.9)$$

$$E_{\rm fin} = \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}\left(n\sqrt{n^2 - 4\Big|m + \frac{1}{2}\beta J\Big|^2} - n^2 + 2\Big|m + \frac{1}{2}\beta J\Big|^2\right). \quad (8.10)$$

It is gratifying to see that in the $\sigma_d \to 0$ limit this result matches our expectation (8.1).

According to the discussion in section 3 these corrections should be compared to the finite size contributions to the energy of spin chain eigenstates. We now discuss their calculation following the same steps as in the undeformed case [27].

8.2 Spin chain side

As should be obvious by now, it should be quite easy to find the $1/J$ corrections to the energies of spin chain states in the presence of a real deformation parameter. This is clear from the discussion of the Bethe ansatz equations (see (6.4) and section 7) that the structure of the subleading terms in the $1/J$ expansion is not affected by the shifts of the mode number by β. Thus, the agreement between the string and gauge theory results found in [27] should go over to the real β deformation case without any modification.

For complex β the situation is less clear and we will consider it in some detail. The conclusion of the previous subsection was that the 1-loop correction to the classical energy of the string solution (8.5) depends on the deformation parameter β only in the combination $|m+\frac{1}{2}\beta J|$. At the same time, the Bethe equations (6.1), (6.2) depend separately on the real and imaginary parts of the deformation parameter; it seems that a small miracle is needed for them to combine to produce the energy as a function of the complex β as in (8.10). Remarkably, this is indeed what happens.

As in section 6, in this subsection we shall use the notation $L \equiv J = J_1 + J_2,\ M = J_2$; the $J_1 = J_2$ case of the previous subsection corresponds to $L = 2M$. The starting point of the calculation is the thermodynamic limit of the Bethe equations and the cyclicity condition (6.1). The leading contribution can be extracted without much complication (6.19). To properly take into account the configuration of roots with spacing of order $1/L$ and find the leading finite size correction to the energy we follow [27] and use an integral representation for the logarithm of the factors in (6.1) depending on the difference of rapidities. Focusing on the solutions with equal mode numbers for all M roots ($n_1 = n_2 \ldots = n_M$ in (6.7)), the Bethe equations then become

$$-2\pi(n_1 + \beta_d L) + \frac{1}{x_k} = 2\sum_{j\neq k=1}^{M}\int_0^{1/L}\frac{f(x_k, x_j)\,d\epsilon}{\epsilon^2 + (f(x_k, x_j))^2}\,, \quad (8.11)$$

where on the left hand side we kept the first two nontrivial orders in the $1/L$ expansion and introduced the function $f(x_k, x_j)$

$$f(x_k, x_j) = \frac{x_k - x_j}{1 + K^2 x_k x_j}\,, \qquad K \equiv 2\pi\kappa_d L\,, \tag{8.12}$$

where K is kept finite in the limit $L \to \infty$. From (8.11) it is clear that the change of variables (6.16) isolates the difference between the deformed (XXZ chain) and undeformed (XXX chain) cases and making it easy to compare the two calculations.

Once $x_k - x_j$ is of order of $1/L$ the long-chain expansion becomes unreliable as the two terms in the denominator become of the same order and the integral gives an $\mathcal{O}(1)$ contribution to (8.11). To take into account such configurations we define the resolvent

$$G(x) = \frac{1}{L}\sum_{k=1}^{M}\frac{1}{f(x, x_k)} = \frac{1}{L}\sum_{k=1}^{M}\frac{1 + K^2 x x_k}{x - x_k} \to \int d\xi\,\rho(\xi)\,\frac{1 + K^2 x\xi}{x - \xi} \tag{8.13}$$

where we introduced the density of roots $\rho(\xi)$. The density satisfies the condition that $\rho(\xi)d\xi$ is real and positive and normalized to the filling fraction $\alpha = M/L$. As in the case of the XXX chain [27], our goal will be to rewrite (8.11) in terms of the resolvent. Direct comparison with the calculation of [27] can be done at every step by taking the limit of vanishing deformation. Indeed, as $K \to 0$ (8.13) reduces to the resolvent used in [27].

It is important to notice that the resolvent (8.13) we introduced here does not vanish at infinity. Thus, it is not the density of roots but rather a rescaled density of roots which is related to its discontinuity. The expression for the energy in terms of the density is changed compared to (7.23), and this change can be compactly taken into account by expressing the energy directly in terms of G. Then the energy of the spin chain states accessible in the thermodynamic limit and the cyclicity constraints (6.1),(6.2) become:

$$E = \frac{\lambda}{8\pi^2}\frac{1}{L^2}\sum_{k=1}^{M}\frac{1 + K^2 x_k^2}{x_k^2} = -\frac{\lambda}{8\pi^2 L}\,G'(0)\,, \tag{8.14}$$

$$G(0) = -2\pi(m + \beta_d M) = -2\pi(n_1 + \beta_d L)\,\frac{M}{L}\,. \tag{8.15}$$

The first expression for $G(0)$ is obtained from the thermodynamic limit of (6.1) while the second form is obtained by summing up the k equations (8.11). Thus (cf. (6.10))

$$mL = n_1 M\,. \tag{8.16}$$

In particular, for the $J_1 = J_2$ case ($L = 2M$) we have $m = \frac{1}{2}n_1$. As discussed in [27], while on spin chain side one may consider states with arbitrary $m = n_1 M/L$, only states with integer m correspond to operators of $\mathcal{N} = 4$ SYM and thus also string-theory states.

The goal of rewriting (8.11) in terms of $G(x)$ as well as of making a consistent $1/L$ expansion can be achieved by multiplying (8.11) by the generic term in the resolvent and summing over k. After a few simple algebraic manipulations, we find that the content of the equations (8.11) is captured by

$$-2\pi(n_1+\beta_d L)G(x)+K^2\frac{M}{L}\left(1-\frac{M}{L}\right)-G^2(x)+\frac{1}{x}\Big(G(x)-G(0)\Big)= \tag{8.17}$$

$$=\frac{1}{L}(1+K^2x^2)\left[G'(x)-\sum_{\substack{j\neq k\\ j,k=1}}^{M}\frac{1+Kx_kx_j}{(x-x_k)(x-x_j)}\int_0^{1/L}\frac{\epsilon^2\,d\epsilon}{\epsilon^2+(f(x_k,x_j))^2}\right].$$

This equation cleanly separates the terms contributing to finite size corrections from those surviving in the infinite chain limit for any configuration of roots. Indeed, the right hand side is manifestly of order $1/L$. This allows us to find the resolvent perturbatively in $1/L$ by making the ansatz

$$G=G_0+\frac{1}{L}G_1+\cdots. \tag{8.18}$$

Imposing the correct behaviour (8.15) at the origin, the leading order resolvent $G_0(x)$ is given by

$$G_0(x)=\frac{1}{2}\left[\frac{1}{x}-2\pi(n_1+\beta_d L)-\right. \tag{8.19}$$

$$\left.-\frac{1}{x}\sqrt{[1-2\pi(1-2\alpha)(n_1+\beta_d L)\,x\,]^2+4\,\alpha(1-\alpha)|2\pi(n_1+\beta L)|^2\,x^2}\right],$$

where

$$\alpha\equiv\frac{M}{L}.$$

Using (8.14) it is then a simple exercise to derive the energy of this spin configuration

$$E_0=-\frac{\lambda}{8\pi^2L}\,G_0'(0)=\frac{\lambda}{2L}\alpha(1-\alpha)|n_1+\beta L|^2. \tag{8.20}$$

This is the same expression we found for the corresponding Landau-Lifshitz solution (4.25) (where m $=n_1$, $L=J$). Note that while on the "fast string" action side (4.21) it is clear that the energy should depend only on β, the initial form of the Bethe equations obscures this property, which is restored only at the end of the calculation.

Let us now analyze the subleading corrections. The correction G_1 to the leading order resolvent is easily found by plugging (8.18) into the left hand side of (8.17) and keeping only the terms of order $1/L$. In spite of the form of (8.19), $G_1'(0)$ has a simple expression

$$G_1'(0)=G_0'(0)-\sum_{\substack{j\neq k\\ j,k=1}}^{M}\frac{1+K^2x_jx_k}{x_jx_k}\int_0^{1/L}\frac{\epsilon^2\,d\epsilon}{\epsilon^2+(f(x_k,x_j))^2}. \tag{8.21}$$

To do the two summations we first notice that since we need only the leading order contribution of the integral, the only relevant rapidity configurations are those in which x_j-x_k

is of order $\epsilon \sim 1/L$. As in [27] we will approximate this difference by a linear function weighted by the leading-order density of roots

$$x_j - x_k \simeq \frac{j-k}{L\rho_0(x_k)}\,, \tag{8.22}$$

and replace x_k with x_j elsewhere. It is then easy to do the summation over the difference $(j-k)$. In the process, the term $G_0'(0)$ in (8.21) cancels out and we are left with

$$G_1'(0) = -\sum_{k=1}^{M}\frac{1+K^2x_k^2}{x_k^2}\int_0^{1/L} d\epsilon\,\left[\pi L(1+K^2x_k^2)\rho_0(x_k)\epsilon\right]\coth\left[\pi L(1+K^2x_k^2)\rho_0(x_k)\epsilon\right], \tag{8.23}$$

where the factors of $(1+K^2x_k^2)$ under the integral sign come from the denominator of $f(x_j,x_j)$ in (8.21). Changing the integration variable to $\xi = \pi L(1+K^2x_k^2)\rho_0(x_k)\epsilon$ and converting the remaining sum into an integral using the measure defined by (8.22), i.e. $\sum_k 1/(L\rho(x_k)) \to \int dy$, we end up with

$$G_1'(0) = -\frac{1}{\pi}\int\frac{dy}{y^2}\int_0^{\pi(1+K^2y^2)\rho_0(y)} d\xi\,\xi\,\coth\xi\,, \tag{8.24}$$

where the y-integral runs over the support of the density function.

As already mentioned above, the density of roots is usually equal to the discontinuity of the resolvent across its cut, but with our definition of the resolvent (8.13) it is $(1+K^2y^2)\rho_0(y)$ that has this property. $G_0(x)$ found in (8.19) does not have a cut on the real axis, but rather for complex rapidities. Thus, as in the case of the rational solution of the XXX chain case (i.e. the $su(2)$ sector of $\mathcal{N}=4$ SYM), the roots of the Bethe equations do not lie on the real axis. The endpoints of the cut are determined by zeros of the square root in $G_0(x)$ (8.19), and the cut intersects the real axis at some point. Then the density function is

$$\begin{aligned}(1+K^2y^2)\rho_0(y) &= \frac{1}{2\pi i}\Big(G_0(y+i0)-G_0(y-i0)\Big)\\ &= \frac{1}{2\pi i\,y}\sqrt{(4\pi\mathcal{M}\,y)^2+(1-z_0\,y)^2}\,,\end{aligned} \tag{8.25}$$

where

$$\mathcal{M} \equiv \sqrt{\alpha(1-\alpha)}|(n_1+\beta L)|\,, \tag{8.26}$$

and z_0 is related to the intersection point between the support of the root density (the cut) and the real (its value will not be important for us). This density is the same as in the case of the undeformed theory, except for the terms proportional to β under the square root which enter only through $|n_1+\beta L|$. There is no additional dependence on the real or imaginary parts of β in $G_1'(0)$ and in the energy. Thus it is clear that the finite size corrections we find can be obtained from those of the XXX chain by replacing the mode number n_1 by $|n_1+\beta L|$. This is exactly the same as the result of the analysis of the fluctuations of the fast string action in the previous subsection.

Changing the integration variable to $y' = 1/y - z_0$ we bring the endpoints of the support of the root density to the imaginary axis. Since the integrand in (8.24) is an analytic function we can deform the cut so that to place it on the imaginary axis. Then, making explicit that the new integration variable is purely imaginary, $y' = iz$, and also redefining $\xi = -i\zeta$, the integral in (8.24) becomes

$$G_1'(0) = -\frac{1}{\pi}\int_{z_-}^{z_+} dz \int_0^{\frac{1}{2}\sqrt{(4\pi\mathcal{M})^2-z^2}} d\zeta\,\zeta\,\cot\zeta\,, \tag{8.27}$$

where $z_\pm$ are the endpoints of the support of the density function in these coordinates, $z_\pm = \pm 4\pi\mathcal{M}$. Trading the integration over z for an integral over the upper limit of the ζ integral and integrating by parts we finish with the following expression for the derivative of the first finite size correction to the resolvent and thus the energy shift in (8.14)

$$E_1 = -\frac{\lambda}{8\pi^2 L^2}\,G_1'(0) = \frac{2\lambda}{L^2}\,\mathcal{M}^3\int_{-1}^{1} dx\,x\sqrt{1-x^2}\cot\left(2\pi\mathcal{M}x\right).$$

This is the same expression as in [27] with $n_1 \to |n_1 + \beta L|$ in $\mathcal{M}$ defined in (8.26).

Representing the cotangent as $\pi a\cot(\pi a) = -\sum_{n=-\infty}^{\infty}\frac{a^2}{n^2-a^2}$ and integrating the resulting terms we get the same series expression for the correction to the energy as in (8.10) where $m = \frac{1}{2}n_1$ and $L = 2M = J$.

9. Summary and further directions

In this paper we have shown that the agreement between the gauge theory and the string theory calculations of the anomalous dimensions of "long" 2-scalar operators in $\mathcal{N} = 4$ SYM survives certain exactly marginal deformations of the gauge theory. The supergravity background dual to the gauge theory with a real deformation parameter (as defined in (2.1) and (2.2)) was constructed in [34] using a combination of T-duality transformations and an isometric shift. We have argued that the world sheet Green-Schwarz model in this background is integrable.

The relation between this background and $AdS_5 \times S^5$ also implies that classical solutions in the deformed and undeformed theories are mapped onto each other; as an example, we have presented the deformed 2-spin "circular" string solution which is direct counterpart of the one in [8]. We have shown that after properly identifying the parameters on the two sides of the duality, it is possible to match the "fast string" expansion in the deformed background (the analog of the Landau-Lifshitz action) with the continuum limit of the coherent-state action for the corresponding spin chain. Also, the gauge theory spin chain correctly reproduces the "small string" limit of the sigma model (the analog of the BMN limit in $AdS_5\times S^5$). We have also speculated about the 2-loop structure of the dilatation operator in the 2-spin sector. Starting from the sigma model on $\mathbb{R}\times S^3_\beta$ we have derived the string Bethe equations for real deformation parameter and found that they are remarkably close to those in the undeformed background. Finally, we have also analyzed "finite size" corrections to the anomalous dimensions of operators dual to circular rotating strings and found that they continue to reproduce the world sheet predictions even in the presence of the deformation.

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In a nutshell, the correspondence between the gauge theory and the string theory in this deformed, less supersymmetric, but still conformal case works very well — perhaps better than we had the right to expect. Our analysis has a number of interesting extensions discussed below.

In the case of the $AdS_5 \times S^5$ — $\mathcal{N} = 4$ SYM duality an underlying reason for this "semiclassical" agreement is the equivalence of the two integrable models governing the leading order corrections. For a real deformation parameter ($|\mathbf{q}| = 1$ in the notation of section 2) we have seen that the same is still true. In particular, this suggests that there exist many more sigma models which are classically integrable even though their target space is not a coset space. After all, T-duality combined with isometric coordinate shifts locally maps solutions of the original equations of motion to solutions of the dual equations and thus should preserve the property of integrability. Global issues (like twists of boundary conditions which we have encountered in our discussion) need special treatment.

Assuming that integrability survives at the quantum string level (cf. [80]), our observations suggest that the full β-deformed $\mathcal{N} = 4$ SYM theory is integrable for real β. This is supported by the integrability of the 1-loop gauge-theory dilatation operator, i.e. the integrability in the *small* λ region.

Special examples in which this behavior is clearly realized include the case of rational β-parameter, or, more generally, theories related to $\mathcal{N} = 4$ SYM by some sort of orbifold construction. In the orbifold construction some world sheet fields may have twisted boundary conditions. Due to its relation to string theory in $AdS_5 \times S^5$, the corresponding world sheet sigma model is classically integrable, and, quite likely, integrable to all orders in the world sheet perturbation theory (but still at vanishing string coupling g_s). On the gauge theory side, the dilatation operator has the same local structure as in the undeformed theory and is modified only due to "boundary effects" [31]. The single-trace gauge-invariant operators in the orbifolded theory are obtained from those in the original theory by inserting inside the trace the twist operators representing the orbifold action on the gauge degrees of freedom. As a result, the associated dilatation operator can be represented by the same spin chain as in the original theory but with twisted boundary conditions. Then the relation between the gauge and the string theory integrable systems is guaranteed (to all orders in perturbation theory) through the orbifold construction by the relation between the parent theories.

A question suggested by these observations is whether there exist other transformations which act nontrivially on sigma models while preserving integrability. Classically, any symmetry of the space of solutions of the classical equations of motion of an integrable theory has this property. Finding such symmetries, however, is not always straightforward.

For complex deformation parameter β the existence of integrability is not clear; while the 2-scalar field sector enjoys 1-loop integrability for such a deformation, we have mentioned that at two loops integrability appears to be problematic. In the larger 3-spin sector already the 1-loop integrability appears to be absent [37, 38]. On the string side, the lack of integrability may be related to the use of S-duality in the construction of the corresponding background. It would be important to make this precise and clarify further the lack of integrability in the presence of a complex deformation.

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An interesting extension of our work is to the 3-spin sector. While the string sigma model is likely not to be integrable for complex deformation β, it is still possible to construct the "fast string" expansion and the analog of the Landau-Lifshitz action. On the gauge theory side one can also find the coherent state action for the corresponding (*non*-integrable) spin chain and compare the two results. The limit of real deformation (the case in which all the details should be fairly similar to the ones described in this paper) should help to understand how the integrability is restored. It is possible that while both the string theory and the gauge theory lack integrability, the associated "semiclassical" effective actions describing states with large quantum numbers continue to be the same. In the limit of vanishing β-deformation one should recover the effective action describing the $su(3)$ sector of $\mathcal{N}=4$ SYM analyzed in [77, 78].

Another obvious generalization is the analysis of the $sl(2)$ sector of operators $\mathrm{Tr}\, D_+^S \Phi^J + \cdots$. Given the way the deformation parameter enters the gauge theory action (2.6), it is not hard to see that the 1-loop dilatation operator in this sector should be the same as in the undeformed case [22]. On the string theory side the same is obviously true for the leading term of the reduced string sigma model action. The AdS_5 part of the metric is the same, and we have also an S^1 part of the 5-space (the functions H and G appearing in the deformed solution (2.18) depend only on the angles of the 5-sphere and are constant when restricted to configurations from the $sl(2)$ sector). Therefore, the spin chain Landau-Lifshitz action matches the string action as in the undeformed case [78, 79]. At higher loops, the situation is less clear and should be quite interesting to study. On the string theory side, the world sheet sigma model appears unmodified at the classical level, but quantum corrections will feel the deformation parameter. That suggests that on the gauge theory side the dependence on the deformation parameter should drop out of the 2-loop anomalous dimensions to leading order in $1/J$ but should reappear at subleading orders.

A more speculative direction is to try to reverse-engineer (parts of) supergravity duals of gauge theories exhibiting integrability by starting from the spin chain description of the dilatation operator. In the cases when it is possible to construct its coherent state continuum limit, the resulting action can be interpreted as the "fast string" expansion of a sigma model in the geometry we are interested in constructing. The dependence of this action on the large quantum number may hint at where the fast coordinate should be reinstated. While it is unlikely to be possible to undo the "fast string" limit, the result might in principle be used as a starting point for an ansatz for the complete solution. It would be interesting to construct the supergravity dual of the second exactly marginal deformation of $\mathcal{N}=4$ SYM. Another apparently simple deformation on the gauge theory side — which also breaks conformal invariance — is relaxing the relation between the coefficient of the superpotential and the gauge coupling. This will produce an $\mathcal{N}=1$ flow which would be interesting to study from a supergravity perspective.

From a broader perspective, it is hard to underestimate the importance of studying deformations of the $AdS_5 \times S^5$ — $\mathcal{N}=4$ SYM duality. In the past, such efforts led to important advances in our understanding of 4-dimensional field theories. A further requirement that the deformations preserve a (somewhat fragile) integrable structure may lead to finer probes of the duality and, ultimately, to a better understanding of the spectrum of strings in $AdS_5 \times S^5$ and less-supersymmetric backgrounds.

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A. The general spin chain

There exists a relatively large class of supersymmetric deformations of $\mathcal{N}=4$ SYM whose dilatation operators are described by integrable spin chains.

The hamiltonian of a spin chain which includes as special cases the 2- and 3-field sectors of the β-deformed $\mathcal{N}=4$ SYM is [37]

$$H = \sum_{l=1}^{L} H_{l,l+1}\,, \tag{A.1}$$

$$\begin{aligned} H_{l,l+1} = -\frac{\lambda}{8\pi^2}\,\mathcal{F}\,q\Bigg[&\frac{1-\Upsilon^2 q^2}{1-q^2}\sum_i e_l^{ii}\otimes e_{l+1}^{ii} + \frac{q(1-\Upsilon^2)}{1-q^2}\sum_{i\neq j} e^{\alpha_{ij}}\, e_l^{ij}\otimes e_{l+1}^{ji} + \\ &+\Upsilon^2\sum_{i<j} e_l^{ii}\otimes e_{l+1}^{jj} + \sum_{i>j} e_l^{ii}\otimes e_{l+1}^{jj}\Bigg]\,, \end{aligned} \tag{A.2}$$

where $L=J$ is the length of the spin chain.[35] In this expression the indices i and j take as many values as the number of states P at each site of the chain, and e_l^{ij} are the generators of the $GL(P)$ $(e_l^{ij})_{pq} = \delta_p^i\delta_q^j$ acting at site l. α_{ij} is an antisymmetric $P\times P$ matrix. For the case of $P=2$ discussed below it has single component denoted by α.

In section 6 we were interested in the 2-field sector of the β-deformed $\mathcal{N}=4$ SYM. In this case the hamiltonian (A.2) can be put in a more familiar form using the Pauli matrices

$$e^{12} = \boldsymbol{\sigma}^+\,,\qquad e^{21} = \boldsymbol{\sigma}^-\,,\qquad e^{11} = \frac{1}{2}\left(\mathbb{1}+\boldsymbol{\sigma}^z\right),\qquad e^{22} = \frac{1}{2}\left(\mathbb{1}-\boldsymbol{\sigma}^z\right). \tag{A.3}$$

The lagrangian whose 1-loop dilatation operator is the hamiltonian (A.2) (up to the addition of the identity operator and restricted to the two states per site, $P=2$) contains 2 scalar fields Φ_i in adjoint representation of SU(N)

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\,\mathrm{Tr}\Big[&\partial\Phi_i\partial\bar\Phi^i - \mathcal{F}\,q[k|\Phi_1\Phi_2 - W\,\Phi_2\Phi_1|^2 + C\,(\Phi_1\bar\Phi^1+\Phi_2\bar\Phi^2)(\bar\Phi^1\Phi_1+\bar\Phi^2\Phi_2) + \\ &+A\Phi_2\Phi_1\bar\Phi^2\bar\Phi^1 + B\Phi_1\Phi_2\bar\Phi^1\bar\Phi^2]\Big] \end{aligned}\tag{A.4}$$

[35] The parameter Υ was denoted by Δ in [37]. Here we changed the notation to avoid possible confusion with the standard notation for conformal dimension.

where

$$A = kW + \frac{q(1-\Upsilon^2)}{1-q^2} e^{\alpha}\,, \qquad B = k\overline{W} + \frac{q(1-\Upsilon^2)}{1-q^2} e^{-\alpha}\,, \qquad C = \frac{1-\Upsilon^2 q^2}{1-q^2}\,,$$
$$k = -\frac{q^2\,(1-\Upsilon^2)}{1-q^2}\,, \qquad |W|^2 = \frac{1}{q^2}\,. \tag{A.5}$$

The structure of the last two terms implies that for the theory to be unitary the parameter α must be chosen to be imaginary. The lagrangian (A.4) can be coupled to gauge field, and their contribution to (A.2) is trivial.

In the absence of gauge field, the lagrangian (A.4) has four free parameters: the overall coupling constant $g\sqrt{\mathcal{F}qk}$, the ratio C/qk and the real and imaginary parts of A/qk. The equation (A.5) puts them in one to one correspondence with the spin chain parameters q, Υ and α. In the presence of gauge field there is an additional parameter — the gauge coupling — which can be identified with the already present coupling g. However, since the gauge interactions are flavor-blind, it contributes only to rather trivial shifts in the eigenvalues of the hamiltonian. Its value can be adjusted so that the ground state of H has vanishing energy. We will use this additional freedom below.

By adding terms contributing only to nonplanar (and perhaps self-energy) diagrams, the last term on the first line of (A.4) can be written as a perfect square of an imaginary combination. Nevertheless, due to the terms on the second line, the lagrangian (A.4) cannot be supersymmetrized as long as $A \neq 0$ and $B \neq 0$.

The potential term in the lagrangian (A.4) is the same as the potential of the *nonconformal* β-deformed $\mathcal{N}=4$ SYM (2.6) in the 2-field ($\Phi_3 = 0$) sector

$$V = |he^{i\pi\beta}|^2\,\mathrm{Tr}\,[|\Phi_1\Phi_2 - \mathbf{q}\Phi_2\Phi_1|^2] - 2\,\mathrm{Tr}\,[\,(\Phi_1\bar{\Phi}^1 + \Phi_2\bar{\Phi}^2)(\bar{\Phi}^1\Phi_1 + \bar{\Phi}^2\Phi_2)]\,, \tag{A.6}$$

provided we make the following identifications (using the notation in (2.3) and (2.6))

$$A = 0\,, \qquad B = 0 \quad \Rightarrow \quad W = \frac{e^{-2\pi i\beta_d}}{q}\,, \qquad \alpha = -2\pi i\beta_d\,, \qquad \beta_d \in \mathbb{R} \tag{A.7}$$
$$\mathbf{q} = W\,; \qquad q = \frac{1}{q_d} = e^{2\pi\kappa_d}\,; \qquad q\,k\,\mathcal{F} = |he^{i\pi\beta}|^2\,; \qquad -\frac{C}{k} = \frac{1-\Upsilon^2 q^2}{q^2(1-\Upsilon^2)} = \frac{2}{|he^{i\pi\beta}|^2}\,.$$

Solving these conditions implies that the coefficients appearing in this special case of (A.2) are:

$$\Upsilon^2 = \frac{2 - q_d^2|he^{i\pi\beta}|^2}{2-|he^{i\pi\beta}|^2}\,, \qquad \frac{1-\Upsilon^2 q^2}{1-q^2} = \frac{2}{2-|he^{i\pi\beta}|^2}\,, \qquad \frac{q^2(1-\Upsilon^2)}{1-q^2} = \frac{|he^{i\pi\beta}|^2}{2-|he^{i\pi\beta}|^2}\,, \tag{A.8}$$

Then, expressing (A.2) in terms of the Pauli matrices (A.3) and using the freedom of adding the indentity operator to cancel the vacuum energy, the 2-spin interaction hamiltonian is easily found to be

$$H_2 = -\frac{|h|^2\lambda}{16\pi^2}\Big[\cos 2\pi\beta_d\,(\boldsymbol{\sigma}^x\otimes\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^y\otimes\boldsymbol{\sigma}^y) + \cosh 2\pi\kappa_d\,\,(\boldsymbol{\sigma}^z\otimes\boldsymbol{\sigma}^z - \mathbb{1}\otimes\mathbb{1}) +$$
$$+ \sin 2\pi\beta_d\,(\boldsymbol{\sigma}^x\otimes\boldsymbol{\sigma}^y - \boldsymbol{\sigma}^y\otimes\boldsymbol{\sigma}^x) + \sinh 2\pi\kappa_d\,(\boldsymbol{\sigma}^z\otimes 1 - 1\otimes\boldsymbol{\sigma}^z)\Big]\,. \tag{A.9}$$

Summing over all sites and requiring periodic boundary conditions, the last term in (A.9) disappears and we find that the hamiltonian of the spin chain[36] describing the critical β-deformed theory is

$$H = -\frac{|h|^2\lambda}{16\pi^2}\sum_{l=1}^{L}\Big[\cosh 2\pi\kappa_d\left(\boldsymbol{\sigma}_l^z\otimes\boldsymbol{\sigma}_{l+1}^z - \mathbb{1}_l\otimes\mathbb{1}_{l+1}\right) + \cos 2\pi\beta_d\,(\boldsymbol{\sigma}_l^x\otimes\boldsymbol{\sigma}_{l+1}^x + \\ + \boldsymbol{\sigma}_l^y\otimes\boldsymbol{\sigma}_{l+1}^y) + \sin 2\pi\beta_d\,(\boldsymbol{\sigma}_l^x\otimes\boldsymbol{\sigma}_{l+1}^y - \boldsymbol{\sigma}_l^y\otimes\boldsymbol{\sigma}_{l+1}^x)\Big]\,. \tag{A.10}$$

This is the hamiltonian for an XXZ spin chain with broken parity invariance.

The 3-field sector of the β-deformed SYM is described by a particular case of (A.2) only if $q_d = 1$ [37]. It was argued in [38] that no integrable spin chain description exists for the dilatation operator in the presence of a general deformation $q_d \in \mathbb{R}$.

B. Asymptotics of the monodromy matrix

Here we shall provide some details about asymptotics of the monodromy matrix and quasi-momentum used in section 7.

The Lax connection $\mathcal{R}_1$ is not convenient to analyze the asymptotic behavior of the quasi-momentum at infinity, because it does not vanish at large values of the spectral parameter x. To study the asymptotics it is useful to make an inverse gauge transformation with the matrix $\mathcal{M}^{-1}$, and use a nonlocal and nonperiodic Lax connection (7.8) with the field g depending on $\tilde{\tilde{\phi}}_i$ which satisfy the twisted boundary conditions (7.3). Since the matrix $\mathcal{M}$ is not periodic, the monodromy matrix $\mathrm{T}(x)$ is not similar to the path-ordered exponential of the Lax connection (7.8) but is related to it as follows

$$\mathrm{T}(x) = \mathcal{M}(2\pi)\,\mathcal{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\;\mathcal{A}_1(x)\,\mathcal{M}^{-1}(0)\,. \tag{B.1}$$

Therefore, the quasi-momentum has the following representation in terms of the twisted Lax connection (7.8)

$$\begin{aligned} 2\cos p(x) &= \mathrm{Tr}\left(\mathcal{M}_R\cdot\mathcal{P}\exp\int_0^{2\pi}\mathrm{d}\sigma\,\mathcal{A}_1(x)\right), \\ \mathcal{A}_1(x) &= \frac{R_1}{x^2-1} + \frac{x\,R_0}{x^2-1}\,, \end{aligned} \tag{B.2}$$

where the matrix $\mathcal{M}_R$ is given by

$$\mathcal{M}_R = \mathcal{M}^{-1}(0)\,\mathcal{M}(2\pi) = \begin{pmatrix} e^{i\pi\gamma L} & 0 \\ 0 & e^{-i\pi\gamma L}\end{pmatrix}, \qquad L \equiv J = J_1 + J_2\,. \tag{B.3}$$

[36] It turns out that this chain has the affine quantum symmetry $\widehat{\mathrm{SU}(2)}_q$. In [72] this hamiltonian was constructed in the standard way from the generators of this algebra. Translating to our notation, q_d is related to the quantum deformation while β_d is related to the central extension.

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To find the asymptotic behavior of the quasi-momentum at infinity we expand the path-ordered exponential in (B.2) in powers of $1/x$

$$\begin{aligned}\mathrm{Tr}\Bigg[\mathcal{M}_R\Bigg(1+\frac{i\boldsymbol{\sigma}_a}{x}\int_0^{2\pi}\mathrm{d}\sigma R_0^{(a)}+\frac{i\boldsymbol{\sigma}_a}{x^2}\int_0^{2\pi}\mathrm{d}\sigma R_1^{(a)}-\\ -\frac{\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b}{x^2}\int_0^{2\pi}\mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\sigma' R_0^{(a)}(\sigma)R_0^{(b)}(\sigma')+\cdots\Bigg)\Bigg],\end{aligned}\tag{B.4}$$

where we used the representation $R_i=i\boldsymbol{\sigma}_a R_i^{(a)}$. Taking into account that the matrix $\mathcal{M}_R$ is diagonal, we get the following expression

$$\begin{aligned}2\cos p(x) &= 2\cos(\pi\gamma L)-\frac{2}{x}\sin(\pi\gamma L)\int_0^{2\pi}\mathrm{d}\sigma R_0^{(3)}-\frac{1}{x^2}\cos(\pi\gamma L)\left(\int_0^{2\pi}\mathrm{d}\sigma R_0^{(3)}\right)^2-\\ &-\frac{2}{x^2}\sin(\pi\gamma L)\left[\int_0^{2\pi}\mathrm{d}\sigma R_1^{(3)}-\int_0^{2\pi}\mathrm{d}\sigma\mathrm{d}\sigma'\ \mathrm{sign}(\sigma-\sigma')R_0^{(1)}(\sigma)R_0^{(2)}(\sigma')\right]-\\ &-\frac{1}{x^2}\cos(\pi\gamma L)\left[\left(\int_0^{2\pi}\mathrm{d}\sigma R_0^{(1)}\right)^2+\left(\int_0^{2\pi}\mathrm{d}\sigma R_0^{(2)}\right)^2\right]+\cdots,\end{aligned}\tag{B.5}$$

Taking into account that (see, e.g. [17])

$$\int_0^{2\pi}\mathrm{d}\sigma R_0^{(3)}=2\pi\frac{2M-L}{\sqrt{\lambda}}\,,\qquad M\equiv J_2\,,\tag{B.6}$$

we find that the first line of (B.5) is just an expansion of

$$2\cos\left(\pi\gamma L+2\pi\frac{2M-L}{\sqrt{\lambda}\,x}\right).\tag{B.7}$$

Since the quasi-momentum is conserved, the sum of the remaining terms in (B.5) gives a conserved nonlocal integral of motion $C_\infty(\gamma)$.

In the undeformed model on $R\times S^3$, $C_\infty(0)$ is equal to the sum of squares of off-diagonal components of the nonabelian charge $\int R_0$, and it is convenient to set it to 0, so that a classical string solution would be dual to an operator from the holomorphic $su(2)$ subsector of $\mathcal{N}=4$ SYM. In fact, in the undeformed case, one could set the off-diagonal charges to any values because any string solution with nonvanishing off-diagonal charges can be transformed to a string solution with the vanishing off-diagonal charges by using the SO(4) symmetry of S^3.

In the γ-deformed model the SO(4) symmetry is broken, and the degeneracy of the undeformed model is lifted, and, therefore, different values of the integral of motion $C_\infty(\gamma)$ correspond to nonequivalent solutions of the sigma model on $R\times S^3_\gamma$. It is natural to assume that the class of string solutions dual to operators from the holomorphic $su(2)_\gamma$ subsector of the γ-deformed SYM theory is still singled out by the requirement $C_\infty(\gamma)=0$.

Imposing this condition, we get the large x asymptotics of the quasi-momentum

$$p(x) = \pi\gamma L - \frac{2\pi(L-2M)}{\sqrt{\lambda}x} + \cdots\,, \qquad x \to \infty\,. \tag{B.8}$$

The sign was chosen so that to be compatible with the one used in [17].

The asymptotic behavior of the quasi-momentum at zero can be found by expanding the twisted Lax connection in powers of x

$$\begin{aligned} \partial_1 - \mathcal{A}_1(x) &= \partial_1 + R_1 + xR_0 + x^2 R_1 + \cdots = g^{-1}\left(\partial_1 + xL_0 + x^2 L_1\right) g + \cdots \\ &\equiv g^{-1}\left(\partial_1 - \mathcal{L}_1(x)\right) g\,, \end{aligned} \tag{B.9}$$

where $L_\alpha = \partial_\alpha g g^{-1}$ is the left current. Then the quasi-momentum $p(x)$ takes the form

$$\begin{aligned} 2\cos p(x) &= \mathrm{Tr}\left(\mathcal{M}_R\, g^{-1}(2\pi)\,\mathcal{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\; \mathcal{L}_1(x)\, g(0)\right) \\ &= \mathrm{Tr}\left(\mathcal{M}_L \cdot \mathcal{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\; \mathcal{L}_1(x)\right), \end{aligned} \tag{B.10}$$

where the matrix $\mathcal{M}_L$ is given by [41]

$$\mathcal{M}_L = g(0)\,\mathcal{M}_R\, g^{-1}(2\pi) = \begin{pmatrix} e^{i\pi\gamma(L-2M)} & 0 \\ 0 & e^{-i\pi\gamma(L-2M))} \end{pmatrix}. \tag{B.11}$$

Expanding the path-ordered exponential in powers of x, we get

$$\begin{aligned} 2\cos p(x) &= 2\cos(\pi\gamma(L-2M)) + 2x\sin(\pi\gamma(L-2M)) \int_0^{2\pi} \mathrm{d}\sigma L_0^{(3)} - \\ &\quad -x^2\cos(\pi\gamma(L-2M)) \left(\int_0^{2\pi} \mathrm{d}\sigma L_0^{(3)}\right)^2 - \\ &\quad -2x^2\sin(\pi\gamma(L-2M)) \Big[-\int_0^{2\pi} \mathrm{d}\sigma L_1^{(3)} - \\ &\qquad\qquad - \int_0^{2\pi} \mathrm{d}\sigma \mathrm{d}\sigma'\; \mathrm{sign}(\sigma-\sigma') L_0^{(1)}(\sigma) L_0^{(2)}(\sigma') \Big] - \\ &\quad -x^2\cos(\pi\gamma(L-2M)) \left[\left(\int_0^{2\pi} \mathrm{d}\sigma L_0^{(1)}\right)^2 + \left(\int_0^{2\pi} \mathrm{d}\sigma L_0^{(2)}\right)^2 \right] + \cdots\,. \end{aligned} \tag{B.12}$$

Taking into account that [17]

$$\int_0^{2\pi} \mathrm{d}\sigma L_0^{(3)} = -2\pi \frac{L}{\sqrt{\lambda}}\,, \tag{B.13}$$

and setting the nonlocal conserved integral of motion given by the sum of the terms in the second and third lines of (B.12), we get the small x asymptotics of $p(x)$

$$p(x) = \pi\gamma(L-2M) + \frac{2\pi L}{\sqrt{\lambda}}\, x - 2\pi m + \cdots\,, \qquad x \to 0\,, \tag{B.14}$$

where m is an integer quasi-momentum number.

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